MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics

IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1

Exercise 1.1 (British Mathematical Olympiad Round 1 2000 P2)**.** Show that, for every positive integer n, the integer

$$121^n - 25^n + 1900^n - (-4)^n$$

is divisible by 2000.

> **Walkthrough** —
> **(a)**

Solution 1. Let n be a positive integer. Note that

$$\begin{aligned} 121^n - 25^n + 1900^n - (-4)^n &\equiv (-4)^n - 25^n + 25^n - (-4)^n \pmod{125} \\ &\equiv 0 \pmod{125}. \end{aligned}$$

Also note that

$$\begin{aligned} 121^n - 25^n + 1900^n - (-4)^n &\equiv 9^n - 9^n + (-20)^n - (-20)^n \pmod{16} \\ &\equiv 0 \pmod{16}. \end{aligned}$$

Since 125 and 16 are coprime, it follows that

$$121^n - 25^n + 1900^n - (-4)^n$$

is divisible by 2000 for all positive integers n. ∎

Exercise 1.2 (British Mathematical Olympiad Round 1 2011/12 P1)**.** Find all (positive or negative) integers n for which $n^2 + 20n + 11$ is a perfect square. *Remember that you must justify that you have found them all.*

Walkthrough —

(a)

Solution 2. Let n be an integer such that $n^2 + 20n + 11$ is a perfect square. Then there exists a nonnegative integer m such that

$$n^2 + 20n + 11 = m^2.$$

Note that

$$(n + 10)^2 - 89 = m^2$$

holds, which yields

$$(n + 10 - m)(n + 10 + m) = 89.$$

Since 89 is prime and the integer m is nonnegative, it follows that

$$(n + 10 - m, n + 10 + m)$$

is equal to one of $(1, 89)$ or $(-89, -1)$. In the first case, we have

$$n = 35, m = 44,$$

and in the second case, we have

$$n = -55, m = 44.$$

Note that both $n = 35$ and $n = -55$ satisfy the condition that $n^2 + 20n + 11$ is a perfect square. Therefore, the integers n for which $n^2 + 20n + 11$ is a perfect square are

$$35, -55.$$

■

Exercise 1.3 (British Mathematical Olympiad Round 2 2015/16 P4)**.** Suppose that p is a prime number and that there are different positive integers u and v such that p^2 is the mean of u^2 and v^2. Prove that $2p - u - v$ is a square or twice a square.

Walkthrough —

(a)

Solution 3. Since p^2 is the mean of u^2 and v^2, we have

$$u^2 + v^2 = 2p^2.$$

Multiplying both sides by 2, we obtain

$$(u-v)^2 + (u+v)^2 = (2p)^2.$$

This yields

$$(2p-u-v)(2p+u+v) = (u-v)^2.$$

Let n be an odd positive integer dividing both $2p-u-v$ and $2p+u+v$. Then n divides their sum $4p$ and their difference $2(u+v)$. Since n is odd, it divides p and $u+v$. As p is prime, we have $n=1$ or $n=p$.

Let us consider the case that $n=p$. Note that p divides $u+v$ and $u-v$. Since p is odd, it divides both u and v. Since u and v are distinct positive integers, this implies that

$$u^2 + v^2 > 2p^2,$$

which is a contradiction. This shows that $n=1$, and hence, $2p-u-v$ and $2p+u+v$ admit no common odd positive integer divisor other than 1. It follows that there are nonnegative integers a and b, and relatively prime odd positive integers m and n, such that

$$2p-u-v = 2^a m, \quad 2p+u+v = 2^b n.$$

Since $2^{a+b}mn$ is a perfect square, we have that $a+b$ is even and both m and n are perfect squares. This proves that $2p-u-v$ is a square or twice a square. ■

Exercise 1.4 (British Mathematical Olympiad Round 1 2012/13 P4)**.** Find all positive integers n such that $12n-119$ and $75n-539$ are both perfect squares.

> **Walkthrough** —
>
> **(a)**

Solution 4. Let n be a positive integer such that $12n-119$ and $75n-539$ are both perfect squares. Let a, b be nonnegative integers such that

$$12n - 119 = a^2, \quad 75n - 539 = b^2.$$

This yields

$$\begin{aligned}(5a)^2 - (2b)^2 &= 5^2 \cdot 12n - 5^2 \cdot 119 - 2^2 \cdot 75n + 2^2 \cdot 539 \\ &= 2^2 \cdot 420 - 119 \cdot (25-4) \\ &= 21 \cdot (80 - 119) \\ &= -21 \cdot 39 \\ &= -3^2 \cdot 7 \cdot 13,\end{aligned}$$

and this gives

$$(2b-5a)(2b+5a)=3^2\cdot 7\cdot 13.$$

Note that the factors on the left-hand side are both positive integers differing by a multiple of 10. Also note that $3, 13$ are congruent to $3 \pmod{10}$, and 7 is congruent to $-3 \pmod{10}$. This shows that $(2b-5a, 2b+5a)$ is equal to one of the following pairs:

$$(3, 3\cdot 7\cdot 13), (7, 3^2\cdot 13), (13, 3^2\cdot 7), (3\cdot 7\cdot 13, 3), (3^2\cdot 13, 7), (3^2\cdot 7, 13).$$

Since a is nonnegative, we have $2b+5a \geq 2b-5a$, so we only need to consider the first three cases. In the first case, note that 3 divides $10a$, and hence 3 divides a. In the second case, we have $a = 11$. In the third case, we have $a = 5$. Since $12n - 119$ is equal to a^2, and 3 does not divide 119, we have that a is not divisible by 3, and hence a is equal to one of $11, 5$. If $a = 11$, then

$$n = \frac{a^2+119}{12} = \frac{11^2+119}{12} = \frac{240}{12} = 20,$$

and if $a = 5$, then

$$n = \frac{a^2+119}{12} = \frac{5^2+119}{12} = \frac{144}{12} = 12.$$

Observe that

$$12\cdot 20 - 119 = 121 = 11^2,$$

and

$$75\cdot 20 - 539 = 1500 - 539 = 961 = 31^2$$

hold. Also note that

$$12\cdot 12 - 119 = 25 = 5^2,$$

and

$$75\cdot 12 - 539 = 900 - 539 = 361 = 19^2.$$

This proves that $n = 12, 20$ are the only positive integers such that both $12n - 119$ and $75n - 539$ are perfect squares. ■

Exercise 1.5 (USAJMO 2015 P2, AoPS)**.** Solve in integers the equation

$$x^2+xy+y^2 = \left(\frac{x+y}{3}+1\right)^3.$$

Walkthrough —

(a)

Solution 5. Let x, y be integers satisfying the equation. Rearranging, we have

$$(x+y)^2 - xy = \left(\frac{x+y}{3} + 1\right)^3.$$

Note that 3 divides $x+y$. Write $x+y = 3n$ for some integer n. We obtain

$$9n^2 - xy = (n+1)^3,$$

which yields

$$9n^2 - x(3n-x) = n^3 + 3n^2 + 3n + 1,$$

which simplifies to

$$x^2 - 3nx = n^3 - 6n^2 + 3n + 1.$$

This gives

$$4x^2 - 12nx + 9n^2 = 4n^3 - 15n^2 + 12n + 4,$$

which shows that

$$(2x - 3n)^2 = (n-2)(4n^2 - 7n - 2),$$

and hence,

$$(2x-3n)^2 = (n-2)^2(4n+1)$$

holds. This implies that $4n+1$ is a perfect square. Write $4n+1 = (2m+1)^2$ for some integer m. This gives $n = m^2 + m$, and hence we obtain

$$(2x-3n)^2 = (n-2)^2(2m+1)^2,$$

which implies that $2x - 3n$ equals $\pm(n-2)(2m+1)$. This shows that

$$\begin{aligned} x &= \frac{3n + n(2m+1)}{2} - 2m - 1 \\ &= 2n + mn - 2m - 1 \\ &= 2m^2 + 2m + m^3 + m^2 - 2m - 1 \\ &= m^3 + 3m^2 - 1, \end{aligned}$$

holds or

$$\begin{aligned} x &= \frac{3n - n(2m+1)}{2} + 2m + 1 \\ &= n - mn + 2m + 1 \\ &= m^2 + m - m^3 - m^2 + 2m + 1 \\ &= -m^3 + 3m + 1 \end{aligned}$$

holds. It follows that (x, y) is equal to one of

$$(m^3 + 3m^2 - 1, -m^3 + 3m + 1), \quad (-m^3 + 3m + 1, m^3 + 3m^2 - 1).$$

Also note that if m, n are integers satisfying $n = m^2 + m$, and (x, y) is a pair of integers satisfying $x + y = 3n$ and

$$(2x - 3n)^2 = (n-2)^2(2m+1)^2,$$

then it follows that

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

Consequently, the integer solutions to the equation are exactly the pairs of the form

$$(m^3 + 3m^2 - 1, -m^3 + 3m + 1), \quad (-m^3 + 3m + 1, m^3 + 3m^2 - 1),$$

for some integer m. ■

Exercise 1.6 (British Mathematical Olympiad Round 1 2016/17 P6)**.** Consecutive positive integers $m, m+1, m+2, m+3$ are divisible by consecutive odd positive integers $n, n+2, n+4, n+6$ respectively. Determine the smallest possible value of m in terms of n.

Walkthrough —

(a)

Solution 6.

Claim — Let m be a positive integer and n be a positive odd integer such that the integers $m, m+1, m+2$ and $m+3$ are divisible by $n, n+2, n+4$ and $n+6$ respectively. Then m is at least as large as

$$\left(\frac{(n+4)(n+6)+1}{2} + \left(\frac{\mathfrak{n}-1}{2}\right)(n+4)(n+6)\right)(n+2) - 1,$$

where

$$\mathfrak{n} = \begin{cases} n & \text{if } 3 \text{ does not divide } n, \\ \frac{n}{3} & \text{if } 3 \text{ divides } n. \end{cases}$$

Proof of the Claim. Let k, r, s, t be positive integers such that

$$m = kn, \quad m+1 = r(n+2), \quad m+2 = s(n+4), \quad m+3 = t(n+6).$$

Using that n divides m, it follows that n divides $2r - 1$.

Since $n+4$ divides $m+2$, it follows that $n+4$ divides $2r-1$. Finally, using that $n+6$ divides $m+3$, we obtain that $n+6$ divides $2r-1$.

Hence, the integer $2r-1$ is divisible by each of $n, n+4, n+6$. Since $n+4, n+6$ are consecutive odd integers, they are coprime. Thus, $(n+4)(n+6)$ divides $2r-1$. This shows that

$$r = \frac{(n+4)(n+6)+1}{2} + \alpha(n+4)(n+6)$$

for some non-negative integer α. Since n divides $2r-1$, it follows that n divides $24+48\alpha$. Using that n is odd, we obtain that n divides $3(2\alpha+1)$. Write

$$\mathfrak{n} = \begin{cases} n & \text{if 3 does not divide } n, \\ \frac{n}{3} & \text{if 3 divides } n. \end{cases}$$

Note that $\mathfrak{n}$ is odd and $\mathfrak{n}$ divides $2\alpha+1$. Thus, there exists a non-negative integer β such that

$$\alpha = \mathfrak{n}\beta + \frac{\mathfrak{n}-1}{2}.$$

Substituting this into the expression for r, we obtain

$$r = \frac{(n+4)(n+6)+1}{2} + \left(\mathfrak{n}\beta + \frac{\mathfrak{n}-1}{2}\right)(n+4)(n+6).$$

□

Claim — Let R denote the positive integer

$$\frac{(n+4)(n+6)+1}{2} + \left(\frac{\mathfrak{n}-1}{2}\right)(n+4)(n+6),$$

and let M denote the positive integer

$$R(n+2)-1.$$

Then, the integers $M, M+1, M+2, M+3$ are divisible by $n, n+2, n+4, n+6$ respectively.

Proof of the Claim. Note that

$$\begin{aligned} 2R-1 &\equiv 24 + 24(\mathfrak{n}-1) \pmod n \\ &\equiv 24\mathfrak{n} \pmod n \\ &\equiv 0 \pmod n. \end{aligned}$$

This shows that n divides M. Also note that $n+2$ divides $M+1$. Moreover,

$$\begin{aligned} &M+2 \\ &= R(n+2)+1 \end{aligned}$$

$$
\begin{aligned}
&= \frac{(n+2)(n+4)(n+6)+n+4}{2} + \left(\frac{\mathfrak{n}-1}{2}\right)(n+2)(n+4)(n+6) \\
&= \frac{(n+2)(n+6)+1}{2}(n+4) + \left(\frac{\mathfrak{n}-1}{2}\right)(n+2)(n+4)(n+6)
\end{aligned}
$$

holds, which shows that $n+4$ divides $M+2$. Similarly, it follows that

$$
M+3 = \frac{(n+2)(n+4)+1}{2}(n+6) + \left(\frac{\mathfrak{n}-1}{2}\right)(n+2)(n+4)(n+6),
$$

which shows that $n+6$ divides $M+3$. This completes the proof of the claim. □

Combining the two claims, it follows that the smallest possible value of m is equal to M, that is, the integer

$$
\frac{(n+2)(n+4)(n+6)+n}{2} + \left(\frac{\mathfrak{n}-1}{2}\right)(n+2)(n+4)(n+6),
$$

which is equal to

$$
\frac{1}{2}\left(n + \mathfrak{n}(n+2)(n+4)(n+6)\right).
$$

■

References

[Che25] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)