MOPSS

8 November 2025

Mathematics Olympiad

Problem Solving Sessions

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

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Exercise 1.1 (Moscow Mathematical Olympiad 2024 Grade 9 P1, by A. Doledenok, V. Retinsky)**.** Let a, b, c, d be nonzero real numbers satisfying

$$\frac{a}{b}+\frac{b}{a}=\frac{c}{d}+\frac{d}{c}.$$

Prove that the product of some two of the numbers a, b, c, d is equal to the product of the remaining two numbers.

Solution 1. Note that

$$\frac{a}{b}-\frac{c}{d}=\frac{d}{c}-\frac{b}{a}$$

holds, which yields

$$\frac{ad-bc}{bd}=\frac{ad-bc}{ac}.$$

If $ad-bc=0$, then we are done. Otherwise, if $ad-bc$ is nonzero, then we obtain that

$$ac=bd.$$

Thus, in either case, the product of some two of the numbers a, b, c, d is equal to the product of the remaining two numbers. ■

Solution 2. Write

$$t=\frac{a}{b}+\frac{b}{a}.$$

Note that $x+\frac{1}{x}=t$ holds for $x=\frac{a}{b}$ and for $x=\frac{c}{d}$. Thus, the numbers $\frac{a}{b}$ and $\frac{c}{d}$ are the roots of the quadratic equation

$$x^2-tx+1=0.$$

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$ holds. Otherwise, if $\frac{a}{b} \neq \frac{c}{d}$, then they are the two distinct roots of the quadratic equation, and by Vieta's formulas, we have

$$\frac{a}{b} \cdot \frac{c}{d} = 1,$$

which yields $ac = bd$. Thus, in either case, the product of some two of the numbers a, b, c, d is equal to the product of the remaining two numbers. ■

Exercise 1.2 (All-Russian Mathematical Olympiad 2014 Grade 9 Day 1 P1, AoPS, by S. Berlov)**.** On a circle, there are 99 positive integers. If a, b are any two neighbouring numbers on the circle, then $a - b$ is equal to 1 or 2 or $\frac{a}{b} = 2$. Prove that there exists a natural number on the circle that is divisible by 3.

Walkthrough —

(a) Show that if none of the numbers on the circle is divisible by 3, then all the numbers on the circle are congruent to either 1 or 2 modulo 3.

(b) Using that 99 is odd, show that it is impossible to arrange 99 numbers on the circle such that none of them is divisible by 3 and any two neighbouring numbers are not congruent modulo 3.

Solution 3. On the contrary, let us assume that none of the numbers on the circle is divisible by 3. Then, each number on the circle is congruent to either 1 or 2 modulo 3. The given conditions imply that any two neighbouring numbers on the circle are not congruent modulo 3. Since there are 99 numbers on the circle and 99 is odd, it is impossible to have such a configuration. ■

Remark. If none of the numbers on the circle is divisible by 3, then using their congruence classes modulo 3 yields a 2-coloring of a 99 cycle such that no two adjacent vertices are of the same color, implying that[a] the cycle is bipartite. This is a contradiction since odd cycles are not bipartite.

[a] It also implies that the odd 99 cycle has chromatic number 2, which is impossible.

Exercise 1.3 (All-Russian Mathematical Olympiad 2024 Grade 10 Day 1 P1, AoPS, by A. Kuznetsov)**.** Let p and q be different prime numbers. We are given an infinite decreasing arithmetic progression in which each of the numbers p^{23}, p^{24}, q^{23} and q^{24} occurs. Show that the numbers p and q also occur in this progression.

Walkthrough — Let d denote the common difference of the arithmetic progression.

(a) Using that d divides $p^{24} - p^{23}$, show that p does not divide d.

(b) Using that d divides $p^{24} - p^{23}$, show that d divides $p - 1$.

(c) Show that p occurs in the arithmetic progression.

Solution 4. Denote the common difference of the arithmetic progression by d. Since p, q are distinct primes, it follows that p does not divide $p^{23} - q^{23}$. Using that d divides $p^{23} - q^{23}$, we conclude that p does not divide d. Note that d also divides $p^{24} - p^{23}$, which implies that d divides $p - 1$. It follows that d divides $p^{23} - p$. Since d is negative, we conclude that p occurs in the arithmetic progression. By a similar argument, it follows that q also occurs in the arithmetic progression. ∎

Exercise 1.4 (Kürschák Competition 1983 P1, AoPS)**.** Rational numbers x, y and z satisfy the equation

$$x^3 + 3y^3 + 9z^3 - 9xyz = 0.$$

Prove that $x = y = z = 0$.

Walkthrough —

(a) Show that if a, b, c are integers satisfying

$$a^3 + 3b^3 + 9c^3 = 9abc,$$

then 3 divides a, and $(b, c, a/3)$ also satisfies the above equation.

(b) If (x, y, z) is a non-trivial integer solution to the given equation with $|x| + |y| + |z|$ minimum, show that x is nonzero, and that $y, z, x/3$ is also a solution to the given equation.

Solution 5. Note that if x, y, z are rational numbers satisfying the given equation, then dx, dy, dz also satisfy the equation for any positive integer d. Hence, it suffices to prove that there are no integer solutions to the given equation other than the trivial solution $x = y = z = 0$.

Claim — If (a, b, c) are integers satisfying

$$x^3 + 3y^3 + 9z^3 = 9xyz,$$

then 3 divides a, and $(b, c, a/3)$ also satisfies the above equation.

Proof of the Claim. Note that 3 divides a^3. Since 3 is a prime, it follows that 3 divides a. Using

$$a^3 + 3b^3 + 9c^3 = 9abc,$$

we obtain

$$b^3 + 3c^3 + 9\left(\frac{a}{3}\right)^3 = 9bc\left(\frac{a}{3}\right).$$

This completes the proof of the claim. □

Let (x, y, z) be a non-trivial integer solution to the given equation with $|x|+|y|+|z|$ minimum. Note that x is nonzero, otherwise, y, z satisfy $y^3+3z^3 = 0$, which is impossible since y, z are integers. By the above claim, it follows that $y, z, x/3$ is also a solution to the given equation. Using that x is nonzero, we obtain

$$|y| + |z| + \left|\frac{x}{3}\right| < |x| + |y| + |z|,$$

which contradicts the minimality of $|x| + |y| + |z|$. This shows that there are no non-trivial integer solutions to the given equation. ■

> **Remark.** The method used in the above solution is known as *infinite descent*. The idea is to show that if there is a non-trivial solution to the given equation, then there is a **smaller** non-trivial solution. This leads to an infinite sequence of smaller and smaller non-trivial solutions, which is impossible for positive integers.

Exercise 1.5 (Hungarian Mathematical Olympiad 2005/06 Specialized Math Schools P2)**.** Denote by $d(n)$ the number of positive divisors of n. Suppose that r and s are positive integers with the property that $d(ks) \geq d(kr)$ for each $k \in \mathbb{N}$. Prove that r divides s.

> **Walkthrough** —
>
> **(a)**

Solution 6. Note that it suffices to consider the case when r is a power of a prime, say $r = p^a$ for some prime p, and some positive integer a. Indeed, if x is a prime power dividing r, then for each positive integer k, we have

$$d(ks) \geq d(kr) \geq d(kx),$$

and if we can show that every prime power dividing r also divides s, then it follows that r divides s.

Observe that $s > 1$ since $d(s) \geq d(p^a) \geq 2$. Write

$$s = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k},$$

where k is a positive integer, $p_1, p_2, \dots, p_k$ are distinct primes, and $\alpha_1, \alpha_2, \dots, \alpha_k$ are positive integers.

Let us first show that $p = p_i$ for some $i \in \{1, 2, \dots, k\}$. Suppose, for the sake of contradiction, that $p \neq p_i$ for all $i \in \{1, 2, \dots, k\}$. Then, taking $k = p^u$ with

$$u = a(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_k + 1) - a,$$

we have

$$d(ks) = (u + 1)(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_k + 1),$$

$$\begin{aligned} d(kr) &= d(p^{u+a}) \\ &= u + a + 1. \end{aligned}$$

Since $u+a+1$ is coprime to $(\alpha_1+1)(\alpha_2+1)\dots(\alpha_k+1)$, it follows that $u+a+1$ divides $u+1$. This is a contradiction since $a \geq 1$. Thus, $p = p_i$ for some $i \in \{1, 2, \dots, k\}$. Renaming indices if necessary, we may assume that $i = 1$. Taking $k = p^u$ with

$$u = a(\alpha_1 + 1)(\alpha_2 + 1)\dots(\alpha_k + 1) - a,$$

we have

$$\begin{aligned} d(ks) &= (u + \alpha_1 + 1)(\alpha_2 + 1)\dots(\alpha_k + 1), \\ d(kr) &= d(p^{u+a}) \\ &= u + a + 1. \end{aligned}$$

Since $u + a + 1$ is coprime to $(\alpha_2 + 1)\dots(\alpha_k + 1)$, it follows that $u + a + 1$ divides $u + \alpha_1 + 1$. This shows that $a \leq \alpha_1$. ■

Exercise 1.6 (Kürschák Competition 2020 P2, AoPS)**.** Find all functions $f\colon \mathbb{Q} \to \mathbb{R}_{\geq 0}$ such that for any two rational numbers x and y, the conditions

$$f(x+y) \leq f(x) + f(y), f(xy) = f(x)f(y)$$

hold, and $f(2) = \frac{1}{2}$.

Walkthrough — Let $f\colon \mathbb{Q} \to \mathbb{R}_{\geq 0}$ be a function satisfying the given conditions.

(a) Show that $f(0) = 0$ and $f(1) = f(-1) = 1$.

(b) Show that for any integer n, $f(n)$ is nonzero.

(c) Using the fact that any positive integer can be written as a sum of distinct powers of 2, show that for any positive integer n, the inequality $f(n) \leq 2$ holds.

(d) Prove that for any positive integer n, the inequality $f(n) \leq 1$ holds.

(e) For any odd positive integer n and for any positive integer m, show that the inequality $1 - f(n)^{2^m} \leq f(n^{2^m} - 1)$ holds, and using divisibility properties by powers of 2, prove that $1 - f(n)^{2^m} \leq \frac{1}{2^m}$ holds.

(f) Deduce that for any odd positive integer n, $f(n) = 1$ holds.

(g) Prove that for any rational number r,

$$f(r) = \begin{cases} 0 & \text{if } r = 0, \\ 2^{-v_2(r)} & \text{if } r \neq 0. \end{cases}$$

Let $f\colon \mathbb{Q} \to \mathbb{R}_{\geq 0}$ be the function defined above. Show that f satisfies the given conditions.

Solution 7. Let $f : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ be a function satisfying the given conditions. Note that

$$f(2) = f(2)f(1) = f(-2)f(-1)$$

holds. Since $f(2)$ is nonzero, it follows that $f(1), f(-1)$ are nonzero. Also note that

$$f(1) = f(1)f(1) = f(-1)f(-1)$$

holds. Using that $f(1), f(-1)$ are nonzero, we obtain $f(1) = 1$ and $f(-1) = 1$. Further, note that

$$f(0) = f(0)f(2) = \frac{1}{2}f(0),$$

which implies $f(0) = 0$. For any integer n, using

$$f(n)f\left(\frac{1}{n}\right) = f(1) = 1,$$

it follows that $f(n)$ is nonzero.

For any positive integer n, writing n as a sum of distinct powers of 2, we obtain

$$f(n) \leq \sum_{k=0}^{\infty} f(2^k) = \sum_{k=0}^{\infty} f(2)^k = \sum_{k=0}^{\infty} \frac{1}{2^k} = 2.$$

It follows that for any positive integer n,

$$f(n)^m = f(n^m) \leq 2$$

holds for any positive integer m. Thus, $f(n) \leq 1$ holds for any positive integer n. Indeed, if $f(n) > 1$ holds for some positive integer n, then we obtain that

$$2 \geq 1 + m(f(1) - 1)$$

holds for any positive integer m, which is a contradiction.

Let n be an odd positive integer, and let m be a positive integer. Note that

$$1 - f(n)^{2^m} = f(1) - f(n)^{2^m} \leq f(1 - n^{2^m}) = f(-1)f(n^{2^m} - 1) = f(n^{2^m} - 1)$$

holds. Using induction, it follows that the integer $n^{2^m} - 1$ is divisible by 2^m, and consequently,

$$f(n^{2^m} - 1) \leq f\left(\frac{n^{2^m} - 1}{2^m}\right) f(2^m) = \frac{1}{2^m} f\left(\frac{n^{2^m} - 1}{2^m}\right) \leq \frac{1}{2^m}$$

holds. This shows that

$$f(n)^{2^m} + \frac{1}{2^m} \geq 1$$

holds. If $f(n) < 1$, then using the above and that $f(n)$ is nonzero, we obtain that

$$\frac{1}{2} \leq 1 - \frac{1}{2^m} \leq f(n)^{2^m} < \frac{1}{1 + 2^m(f(n)^{-1} - 1)}$$

holds for any positive integer m, which is a contradiction. Therefore, for any odd positive integer n, it follows that $f(n) = 1$.

This proves that for any $r \in \mathbb{Q}$,

$$f(r) = \begin{cases} 0 & \text{if } r = 0, \\ \frac{1}{2^k} & \text{if } r = \frac{2^k a}{b} \text{ for some odd integers } a, b \text{ and for some integer } k \geq 0. \end{cases}$$

Note that for any nonzero rational number r, there exists a unique integer k such that $r = \frac{2^k a}{b}$ holds for some odd integers a, b. The integer k is called the 2-adic valuation of r, and is denoted by $v_2(r)$. Therefore, the function f can be expressed as

$$f(r) = 2^{-v_2(r)}$$

for any nonzero rational number r, and $f(0) = 0$.

If $f\colon \mathbb{Q} \to \mathbb{R}_{\geq 0}$ satisfies this property, that is,

$$f(r) = \begin{cases} 0 & \text{if } r = 0, \\ 2^{-v_2(r)} & \text{if } r \neq 0, \end{cases}$$

for any rational number r, then f also satisfies the given conditions. Indeed, $f(2) = \frac{1}{2}$ holds, and for any nonzero rational numbers x, y, we have

$$f(xy) = 2^{-v_2(xy)} = 2^{-v_2(x) - v_2(y)} = 2^{-v_2(x)} 2^{-v_2(y)} = f(x) f(y).$$

If x, y are rational numbers and at least one of them is zero, then the condition $f(xy) = f(x)f(y)$ is immediate.

Let x, y be rational numbers. If $x = 0$ or $y = 0$ or $x + y = 0$, then the condition

$$f(x + y) \leq f(x) + f(y)$$

holds. Using the multiplicative property of f, it suffices to consider the case that x, y are integers. Indeed, if

$$f(kx + ky) \leq f(kx) + f(ky)$$

holds for some positive integer k, then we have

$$f(x + y) = \frac{1}{f(k)} f(kx + ky) \leq \frac{1}{f(k)} (f(kx) + f(ky)) = f(x) + f(y).$$

Thus, it suffices to consider the case that x, y are integers. If one of $x, y, x + y$ is zero, then the inequality $f(x + y) \leq f(x) + f(y)$ holds.

Suppose that $x, y, x+y$ are nonzero. Note that

$$v_2(x+y) \geq \min\{v_2(x), v_2(y)\}$$

holds, which implies that

$$f(x+y) = 2^{-v_2(x+y)} \leq 2^{-\min\{v_2(x), v_2(y)\}} = \max\{f(x), f(y)\} \leq f(x) + f(y)$$

holds. Therefore, the function f satisfies the given conditions.

Hence, the only function $f\colon \mathbb{Q} \to \mathbb{R}_{\geq 0}$ satisfying the given conditions is given by

$$f(r) = \begin{cases} 0 & \text{if } r = 0, \\ 2^{-v_2(r)} & \text{if } r \neq 0, \end{cases}$$

for any rational number r. ■

Exercise 1.7 (All-Russian Mathematical Olympiad 2014 Grade 10 Day 1 P2, AoPS, by O. Podlipsky)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x)^2 \leq f(y)$$

holds for any $x, y \in \mathbb{R}$ with $x > y$. Show that

$$0 \leq f(x) \leq 1$$

holds for any $x \in \mathbb{R}$.

Walkthrough —

(a) Show that $f(x) \geq 0$ for any $x \in \mathbb{R}$.

(b) Using the given inequality, show that for any $x, y \in \mathbb{R}$ with $x > y$, and for any positive integer n, the inequality

$$f(x)^{2^n} \leq f(y)$$

holds.

(c) Use the above to conclude that $f(x) \leq 1$ for any $x \in \mathbb{R}$.

Solution 8. Note that for any $x \in \mathbb{R}$, we have

$$f(x+1)^2 \leq f(x),$$

which shows that $f(x) \geq 0$.

To show that $f(x) \leq 1$ for any $x \in \mathbb{R}$, let us establish the following claim.

Claim — For any $x, y \in \mathbb{R}$ with $x > y$, and for any positive integer n, the inequality

$$f(x)^{2^n} \leq f(y)$$

holds.

Proof of the Claim. Note that the inequality holds for $n = 1$ by hypothesis. Let us assume that n is a positive integer such that the inequality holds for any $x, y \in \mathbb{R}$ with $x > y$. Then, for any $x, y \in \mathbb{R}$ with $x > y$, we have

$$x > \frac{x+y}{2} > y,$$

which implies that

$$f(x)^{2^n} \leq f\left(\frac{x+y}{2}\right) \quad \text{and} \quad f\left(\frac{x+y}{2}\right)^2 \leq f(y).$$

Combining these two inequalities and using that $f(x)$ is nonnegative, we obtain

$$f(x)^{2^{n+1}} \leq f(y).$$

By the principle of mathematical induction, the claim follows. □

Let x be an arbitrary real number. Using the above claim, we obtain

$$f(x)^{2^n} \leq f(x-1)$$

for **any** positive integer n. If $f(x) \leq 1$, then we are done. It remains to consider the case that $f(x) > 1$, which we assume from now on. It follows that

$$f(x-1) \geq 1 + 2^n (f(x) - 1)$$

for **any** positive integer n. This is impossible since $f(x) - 1$ is positive.

Combining the above, we conclude that

$$0 \leq f(x) \leq 1$$

holds for any $x \in \mathbb{R}$. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)