Pythagoras' theorem

Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics

IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

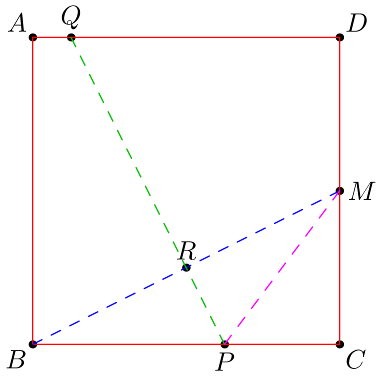

Figure 1: RMO 1990 P3, Exercise 1.1

List of problems and examples

§1 Pythagoras' theorem

Exercise 1.1 (RMO 1990 P3, AoPS)**.** A square sheet of paper $ABCD$ is so folded that B falls on the mid-point M of CD. Prove that the crease will divide BC in the ratio $5 : 3$.

Solution 1. Let the crease intersect BC at P, and AD at Q. Let R denote the midpoint of BM. Then PQ is the perpendicular bisector of BM. Note that $\triangle BPR$ and $\triangle MPR$ are congruent. This gives $BP = PM$. Applying the Pythagoras' theorem to $\triangle CPM$, we obtain

$$CM^2 + CP^2 = PM^2,$$

which gives

$$CM^2 + CP^2 = BP^2.$$

This implies that

$$CM^2 = (BP - CP)(BP + CP),$$

which yields

$$BP - CP = \frac{CM^2}{BP + CP} = \frac{BC}{4} = \frac{BP + CP}{4}.$$

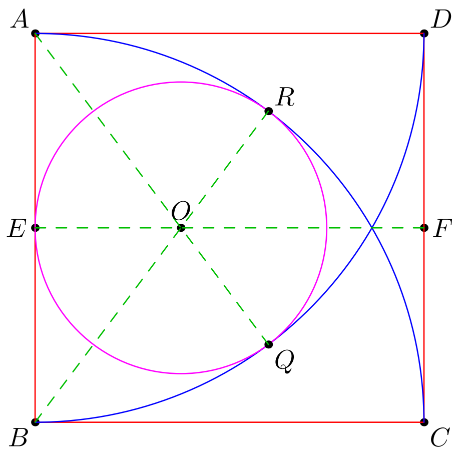

Figure 2: RMO 2012b P1, Exercise 1.2

We get

$$\frac{BP}{CP} = \frac{1+4}{4-1} = \frac{5}{3}.$$

■

Exercise 1.2 (RMO 2012b P1, AoPS). Let $ABCD$ be a unit square. Draw a quadrant of a circle with A as centre and B, D as endpoints of the arc. Similarly, draw a quadrant of a circle with B as centre and A, C as endpoints of the arc. Inscribe a circle Γ touching the arc AC internally, the arc BD internally and also touching the side AB. Find the radius of the circle Γ.

Solution 2. Let E denote the midpoint of AB, and F denote the midpoint of CD. Note that EF is parallel to AB and CD, and has length equal to that of AB and CD. Indeed, the line passing through the midpoint E of AB and parallel to BC bisects AC. Since this line is also parallel to AD, it bisects CD, hence passes through F. Thus, EF is parallel to AB and CD, and has length equal to that of AB and CD.

Note that $\triangle ADF, \triangle BCF$ are congruent, and hence $AF = BF$. This implies that $\triangle AEF, \triangle BEF$ are congruent, and hence EF is a perpendicular bisector of AB.

Let O denote the centre of the circle Γ. Let Q denote the point of tangency of Γ with the arc BD, and R denote the point of tangency of Γ with the arc AC. Since Γ and the arc BD touch at Q, it follows that AQ, QO are perpendicular to their common tangent at Q. Hence, the points A, Q, O are collinear, and this gives

$$AO = AQ - OQ.$$

Similarly, since Γ and the arc AC touch at R, it follows that B, R, O are collinear, and this gives

$$BO = BR - OR.$$

This implies that

$$BR - OR = BC - OR = AD - OR = AQ - OQ = AO.$$

Using an argument as above, it turns out that $\triangle AEO, \triangle BEO$ are congruent, and hence EO is a perpendicular bisector of AB. This implies that O lies on the line EF.

Since Γ is a circle touching AB, it follows that the point of tangency of Γ with AB is the foot of the perpendicular from O to AB. Since OE is perpendicular to AB, it follows that the point of tangency of Γ with AB is E. Thus, the radius of Γ is equal to OE.

Let r denote the radius of Γ, and s denote the length of the sides of $ABCD$. It follows from the above that

$$EO = r, AO = AQ - OQ = s - r.$$

Applying the Pythagoras' theorem to $\triangle AEO$, we obtain

$$r^2 + \left(\frac{s}{2}\right)^2 = (s-r)^2,$$

which gives $r = \frac{3s}{8} = \frac{3}{8}$. ■

Exercise 1.3 (RMO 2012c P1, AoPS)**.** Let $ABCD$ be a unit square. Draw a quadrant of a circle with A as centre and B, D as endpoints of the arc. Similarly, draw a quadrant of a circle with B as centre and A, C as endpoints of the arc. Inscribe a circle Γ touching the arcs AC and BD both externally and also touching the side CD. Find the radius of the circle Γ.

Solution 3. Let E denote the midpoint of AB, and F denote the midpoint of CD. Note that EF is parallel to AB and CD, and has length equal to that of AB and CD. Indeed, the line passing through the midpoint E of AB and parallel to BC bisects AC. Since this line is also parallel to AD, it bisects CD, hence passes through F. Thus, EF is parallel to AB and CD, and has length equal to that of AB and CD.

Note that $\triangle ADF, \triangle BCF$ are congruent, and hence $AF = BF$. This implies that $\triangle AEF, \triangle BEF$ are congruent, and hence EF is a perpendicular bisector of AB. Similarly, it follows that EF is a perpendicular bisector of CD.

Let O denote the centre of the circle Γ. Let Q denote the point of tangency of Γ with the arc BD, and R denote the point of tangency of Γ with the arc AC. Since Γ and the arc BD touch at Q, it follows that AQ, QO are perpendicular to their common tangent at Q. Hence, the points A, Q, O are collinear, and this gives

$$AO = AQ + QO.$$

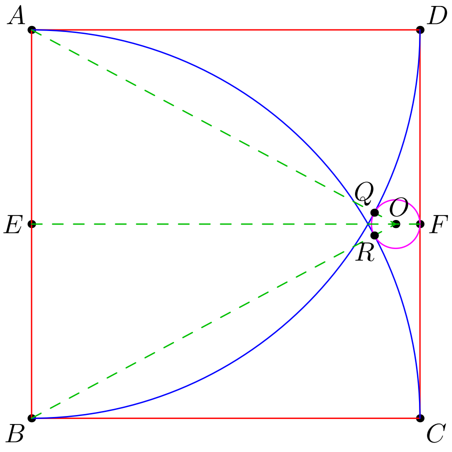

Figure 3: RMO 2012c P1, Exercise 1.3

Similarly, since Γ and the arc AC touch at R, it follows that BR, RO are perpendicular to their common tangent at R. Hence, the points B, R, O are collinear, and this gives

$$BO = BR + RO.$$

This implies that

$$BO = BR + RO = BC + QO = AD + QO = AQ + QO = AO.$$

Using an argument as above, it turns out that $\triangle AEO, \triangle BEO$ are congruent, and hence EO is a perpendicular bisector of AB. This implies that O lies on the line EF.

Since Γ is a circle touching CD, it follows that the point of tangency of Γ with CD is the foot of the perpendicular from O to CD. Since OF is perpendicular to CD, it follows that the point of tangency of Γ with CD is F. Thus, the radius of Γ is equal to OF. This gives

$$EO = EF - OF.$$

Let r denote the radius of Γ, and s denote the length of the sides of $ABCD$. It follows from the above that

$$EO = EF - OF = s - r, AO = AQ + QO = s + r.$$

Applying the Pythagoras' theorem to $\triangle AEO$, we obtain

$$(s-r)^2 + \left(\frac{s}{2}\right)^2 = (s+r)^2,$$

which gives $r = \frac{s}{16} = \frac{1}{16}$. ■

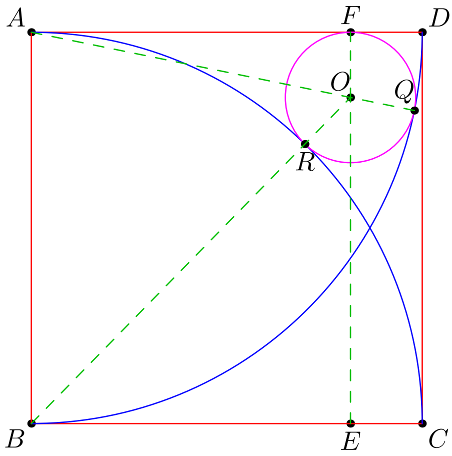

Figure 4: RMO 2012d P1, Exercise 1.4

Exercise 1.4 (RMO 2012d P1, AoPS)**.** Let $ABCD$ be a unit square. Draw a quadrant of a circle with A as centre and B, D as endpoints of the arc. Similarly, draw a quadrant of a circle with B as centre and A, C as endpoints of the arc. Inscribe a circle Γ touching the arc AC externally, the arc BD internally and also touching the side AD. Find the radius of the circle Γ.

Solution 4. Let O denote the centre of the circle Γ, and F denote the point of tangency of Γ with the side AD. Note that OF is perpendicular to AD, and hence OF is parallel to AB. Let E denote the foot of the perpendicular from O to BC. Note that OE is parallel to AB. It follows that E, O, F are collinear, and hence

$$EF = EO + OF.$$

Since EF, AB are parallel and so are E, AF, it follows that $ABEF$ is a parallelogram, and hence, we obtain $EF = AB, BE = AF$.

Let O denote the centre of the circle Γ. Let Q denote the point of tangency of Γ with the arc BD, and R denote the point of tangency of Γ with the arc AC. Since Γ and the arc BD touch at Q, it follows that AQ, QO are perpendicular to their common tangent at Q. Hence, the points A, Q, O are collinear, and this gives

$$AO = AQ - OQ.$$

Similarly, since Γ and the arc AC touch at R, it follows that BR, RO are perpendicular to their common tangent at R. Hence, the points B, R, O are collinear, and this gives

$$BO = BR + RO.$$

Let r denote the radius of Γ, and s denote the length of the sides of $ABCD$. It follows from the above that

$$BO = BR + RO = s + r, AO = AQ - OQ = s - r.$$

Applying the Pythagoras' theorem to $\triangle AOF, \triangle BEO$, we obtain

$$BE^2 + OE^2 = (s + r)^2, AF^2 + OF^2 = (s - r)^2.$$

Using $BE = AF$, we get

$$(OE - OF)(OE + OF) = 4sr,$$

which yields

$$(EF - 2 \cdot OF)(EF) = 4sr.$$

This gives

$$(s - 2r)s = 4sr,$$

implying that

$$r = \frac{s}{6} = \frac{1}{6}.$$

■

Exercise 1.5 (IOQM 2024 P15, AoPS)**.** Let X be the set consisting of twenty positive integers $n, n+2, \ldots, n+38$. The smallest value of n for which any three numbers $a, b, c \in X$, not necessarily distinct, form the sides of an acute-angled triangle is:

Summary — Use Pythagoras' theorem.

Walkthrough —

(a) Use the fact that[a] if a, b, c are the sides of an acute-angled triangle with c denoting the largest side, then $a^2 + b^2 > c^2$ holds.

(b) Show that for any such n, the inequality

$$(n + 38)^2 < 2n^2$$

holds, which yields $n \geq 92$.

(c) Prove that for $n = 92$, the inequality

$$2(n + 2i)^2 > (n + 2i + 2)^2$$

holds for $i = 0, 1, \ldots, 18$.

(d) Note that if suffices to show that

$$2m^2 > (m + 2)^2$$

for any $m \in \{92, 94, \ldots, 128\}$.

What is the role of part (c)? Does it guarantee that for $n = 92$, the set X has the stated property?

[a] How to prove it? Does the **converse** of this statement hold? In other words, if a, b, c are positive real numbers satisfying $a^2 + b^2 > c^2$, then does it follow that a, b, c are the sides of an acute-angled triangle? Even if it doesn't, would it be true under additional hypothesis?

Solution 5. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)