Parallelograms

Mathematics Olympiad

Problem Solving Sessions

MOPSS

DEPARTMENT OF MATHEMATICS
IISER BHOPAL

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Parallelograms

Exercise 1.1 (RMO 2005 P1, AoPS)**.** Let $ABCD$ be a convex quadrilateral. Let P, Q, R, S be the midpoints of AB, BC, CD, DA respectively such that triangles AQR and CSP are equilateral. Prove that $ABCD$ is a rhombus. Determine its angles.

Walkthrough —

(a) Show that PS, RQ are parallel and are of the same length, to conclude that $PQRS$ is a parallelogram.

(b) Using the fact that triangles AQR and CSP are equilateral, show that $\angle CSP, \angle AQR$ are equal.

(c) Use this to prove that AQ is parallel to CS, and are of the same length.

(d) Prove that AB is parallel to CD, and are of the same length, to conclude that $ABCD$ is a parallelogram.

(e) Since the medians AR, CS of $\triangle ACD$ are of the same length, show that $\angle RAC, \angle SCA$ are equal.

(f) Use this to prove that $\triangle RAC, \triangle SCA$ are congruent, and hence, $AD = CD$.

(g) Conclude that $ABCD$ is a rhombus.

(h) Show that the medians of $\triangle ACD$ are equal, to conclude that $\triangle ACD$ is equilateral.

(i) Determine the angles of the rhombus $ABCD$.

Solution 1. Since P, S are mid-points of AB, DA respectively, it follows that PS is parallel to BD and it is equal to $\frac{1}{2}BD$. Similarly, RQ is parallel to BD and it is equal to $\frac{1}{2}BD$. This shows that PS, RQ are equal and parallel to each other.

Thus $\angle CSP, \angle CUQ$ are equal. Since $\triangle CSP, \triangle AQR$ are equilateral, it follows that $\angle CSP, \angle AQR$ are equal to $60°$. This shows that $\angle CUQ$ is equal to $\angle AQR$. Consequently, AQ is parallel to CS, and they are equal. This implies that $AQCS$ is a parallelogram. Hence, AD is parallel to BC, and they are equal since S (resp. Q) is the mid-point of AD (resp. BC). Consequently, $ABCD$ is a parallelogram.

Since the medians AR, CS of $\triangle ACD$ are equal, it follows that $\angle RAC, \angle SCA$ are equal. This implies that $\triangle RAC, \triangle SCA$ are congruent, which gives $AS =$

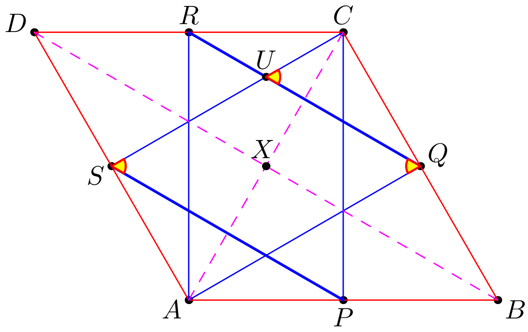

Figure 1: India RMO 2005 P1, Exercise 1.1

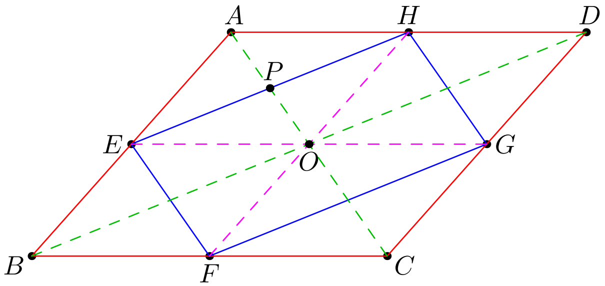

Figure 2: India RMO 2011b P5, Exercise 1.2

RC, and hence, $AD = CD$. Since $ABCD$ is a parallelogram and $AD = CD$, it follows that $ABCD$ is a rhombus.

Denote by X the point of intersection of the diagonals of $ABCD$. Note that DX is a median of $\triangle ACD$, and $DX = \frac{1}{2}BD = PS = CS$. Using the equality of the medians DX, CS of $\triangle ACD$, we get that $AC = AD$. Since AC, AD, CD are equal, it follows that $\triangle ACD$ is equilateral. Thus, $ABCD$ is a rhombus with $\angle ADC = 60°$. Moreover, we have

$$\angle ABC = 60°, \angle DAB = \angle DCB = 120°.$$

■

Exercise 1.2 (RMO 2011b P5, AoPS)**.** Let $ABCD$ be a convex quadrilateral. Let E, F, G, H be the midpoints of AB, BC, CD, DA respectively. If AC, BD, EG, FH concur at a point O, prove that $ABCD$ is a parallelogram.

Walkthrough —

(a) Show that EH, FG are parallel and equal in length, hence $EFGH$ is a parallelogram.

(b) Let P be the point of intersection of AC and EH. Show that the diagonals AO and EH of the quadrilateral $AEOH$ bisect each other.

(c) Show that AE is parallel and equal to OH.

(d) Show that AB is parallel and equal to FH.

(e) Similarly, show that CD is parallel and equal to FG.

(f) Conclude that $ABCD$ is a parallelogram.

Solution 2. Since E, H are the midpoints of AB, AD respectively, it follows that EH is parallel to BD and $EH = \frac{1}{2}BD$. Similarly, considering the midpoints F, G of BC, CD respectively, we get that FG is parallel to BD and $FG = \frac{1}{2}BD$. Thus, EH and FG are parallel and $EH = FG$. Hence, $EFGH$ is a parallelogram, and so, the diagonals EG, FH bisect each other.

Let P be the point of intersection of AC and EH. Since EP is parallel to BO and E is the midpoint of AB, it follows that $AP = PO$. Considering the midpoints H, G of AD, CD respectively, we get that HG is parallel to OP. Since EG, FH bisect each other at O, it follows that $EO = OG$. Combining this with the fact that HG is parallel to OP, we get that $EP = PH$. This shows that the diagonals AO, EH of the quadrilateral $AEOH$ bisect each other. This implies that the triangles $\triangle APE$ and $\triangle OPH$ are congruent. Hence, $AE = OH$ and AE is parallel to OH. Also, since E is the midpoint of AB and O is the midpoint of FH, it follows that AB is parallel to FH and $AB = FH$. Similarly, considering the midpoints F, G of BC, CD respectively, we get that CD is parallel to FG and $CD = FG$. Thus, $ABCD$ is a parallelogram. ■

Remark. The fact that the midpoints of the sides of any quadrilateral form a parallelogram, is known as Varignon's theorem.

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)