Complex numbers

MOPSS

Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Complex numbers

Exercise 1.1 (Moldova National Olympiad 2001 Grade 11 Day 2 P6, AoPS)**.**
For a positive integer n, denote $A_n = \{(x, y) \in \mathbb{Z}^2 \mid x^2 + xy + y^2 = n\}$.

(a) Prove that the set A_n is finite.

(b) Prove that the number of elements of A_n is divisible by 6 for all $n \geq 1$.

(c) For which n is the number of elements of A_n divisible by 12?

> **Remark.** The following walkthrough may be skipped if one does not have an exposure to the notion of **groups**.

Walkthrough —

(a) Note that $x^2 + xy + y^2 \geq \left(x + \frac{y}{2}\right)^2 + \frac{3}{4}y^2$ holds.

(b) For part (b), consider the transformations

$$(x, y) \mapsto (-y, x - y), \quad (x, y) \mapsto (y, x).$$

Show that these transformations[a] map A_n to itself and generate a group of order 6 acting on A_n without fixed points.

(c) For part (c), consider the transformations

$$(x, y) \mapsto (x, y), \quad (x, y) \mapsto (-x, -y).$$

Show that these transformations map A_n to itself, and provides an action of the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on A_n. Determine the fixed points of this action.

[a] Note that

$$(x + y\omega)\omega = x\omega + y\omega^2 = -y + (x - y)\omega$$

holds.

Solution 1. Note that any $(x, y) \in A_n$ satisfies

$$n = x^2 + xy + y^2 = \frac{1}{2}\left((x + y)^2 + x^2 + y^2\right),$$

which implies that $|x|, |y| \leq \sqrt{2n}$. Thus, there are only finitely many pairs (x, y) of integers satisfying $x^2 + xy + y^2 = n$. This proves part (a).

To establish part (b), assume that n is a positive integer. For any $(x, y) \in A_n$, consider the pairs

$$(x, y), (-x-y, x), (y, -x-y),$$

and note that they all belong to A_n. Indeed,

$$(-x-y)^2 + (-x-y)x + x^2 = x^2 + xy + y^2,$$
$$y^2 + y(-x-y) + (-x-y)^2 = x^2 + xy + y^2.$$

Further, if any two of these three pairs were equal, then we would have $x = y = 0$, which is impossible since $n \geq 1$. Thus, the three pairs above are distinct. Hence, the elements of A_n can be grouped into sets of three, proving that $|A_n|$ is divisible by 3. Similarly, for any $(x, y) \in A_n$, the pair (y, x) also belongs to A_n, and is distinct from (x, y) unless $n/3$ is a perfect square. Moreover, for any $(x, y) \in A_n$, the pair $(-y, -x)$ also belongs to A_n, and is distinct from (x, y) unless n is a perfect square. Thus, the elements of A_n can be grouped into sets of two, proving that $|A_n|$ is even. Combining these results, we conclude that $|A_n|$ is divisible by 6.

Finally, to determine when $|A_n|$ is divisible by 12, it suffices to determine when $|A_n|$ is divisible by 4. For any $(x, y) \in A_n$, consider the pairs

$$(x, y), (y, x), (-x, -y), (-y, -x),$$

and note that they all belong to A_n. Moreover, if any two of these four pairs were equal, then we would have either $x = y$ with $n = 3x^2$, or $x = -y$ with $n = x^2$. Therefore, if $n/3$ is not a perfect square and n is not a perfect square, then the four pairs above are distinct, and consequently $|A_n|$ is divisible by 4. Moreover, if $n = 3k^2$ holds for some integer k, then the pairs $(k, k), (-k, -k)$ belongs to A_n, and the remaining elements of A_n can be grouped into sets of four, showing that $|A_n| \equiv 2 \pmod 4$. Similarly, if $n = k^2$ holds for some integer k, then the pairs $(k, -k), (-k, k)$ belongs to A_n, and the remaining elements of A_n can be grouped into sets of four, showing that $|A_n| \equiv 2 \pmod 4$. Thus, $|A_n|$ is divisible by 4 if and only if $n/3$ is not a perfect square and n is not a perfect square. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)