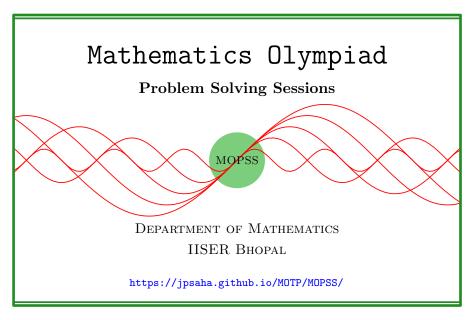
# **Complex numbers**

#### **MOPSS**



### Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [Che25, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

# List of problems and examples

# §1 Complex numbers

**Exercise 1.1** (Moldova National Olympiad 2001 Grade 11 Day 2 P6, AoPS). For a positive integer n, denote  $A_n = \{(x, y) \in \mathbb{Z}^2 \mid x^2 + xy + y^2 = n\}$ .

- (a) Prove that the set  $A_n$  is finite.
- (b) Prove that the number of elements of  $A_n$  is divisible by 6 for all  $n \ge 1$ .
- (c) For which n is the number of elements of  $A_n$  divisible by 12?

**Remark.** The following walkthrough may be skipped if one does not have an exposure to the notion of **groups**.

#### Walkthrough —

- (a) Note that  $x^2 + xy + y^2 \ge (x + \frac{y}{2})^2 + \frac{3}{4}y^2$  holds.
- (b) For part (b), consider the transformations

$$(x,y) \mapsto (-y, x-y), \quad (x,y) \mapsto (y,x).$$

Show that these transformations<sup>a</sup> map  $A_n$  to itself and generate a group of order 6 acting on  $A_n$  without fixed points.

(c) For part (c), consider the transformations

$$(x,y) \mapsto (x,y), \quad (x,y) \mapsto (-x,-y).$$

Show that these transformations map  $A_n$  to itself, and provides an action of the group  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  on  $A_n$ . Determine the fixed points of this action.

<sup>a</sup>Note that

$$(x + y\omega)\omega = x\omega + y\omega^2 = -y + (x - y)\omega$$

holds.

**Solution 1.** Note that any  $(x,y) \in A_n$  satisfies

$$n = x^2 + xy + y^2 = \frac{1}{2} ((x+y)^2 + x^2 + y^2),$$

which implies that  $|x|, |y| \le \sqrt{2n}$ . Thus, there are only finitely many pairs (x, y) of integers satisfying  $x^2 + xy + y^2 = n$ . This proves part (a).

To establish part (b), assume that n is a positive integer. For any  $(x, y) \in A_n$ , consider the pairs

$$(x,y), (-x-y,x), (y,-x-y),$$

and note that they all belong to  $A_n$ . Indeed,

$$(-x-y)^{2} + (-x-y)x + x^{2} = x^{2} + xy + y^{2},$$
  
$$y^{2} + y(-x-y) + (-x-y)^{2} = x^{2} + xy + y^{2}.$$

Further, if any two of these three pairs were equal, then we would have x=y=0, which is impossible since  $n\geq 1$ . Thus, the three pairs above are distinct. Hence, the elements of  $A_n$  can be grouped into sets of three, proving that  $|A_n|$  is divisible by 3. Similarly, for any  $(x,y)\in A_n$ , the pair (y,x) also belongs to  $A_n$ , and is distinct from (x,y) unless n/3 is a perfect square. Moreover, for any  $(x,y)\in A_n$ , the pair (-y,-x) also belongs to  $A_n$ , and is distinct from (x,y) unless n is a perfect square. Thus, the elements of  $A_n$  can be grouped into sets of two, proving that  $|A_n|$  is even. Combining these results, we conclude that  $|A_n|$  is divisible by 6.

Finally, to determine when  $|A_n|$  is divisible by 12, it suffices to determine when  $|A_n|$  is divisible by 4. For any  $(x,y) \in A_n$ , consider the pairs

$$(x,y),(y,x),(-x,-y),(-y,-x),$$

and note that they all belong to  $A_n$ . Moreover, if any two of these four pairs were equal, then we would have either x=y with  $n=3x^2$ , or x=-y with  $n=x^2$ . Therefore, if n/3 is not a perfect square and n is not a perfect square, then the four pairs above are distinct, and consequently  $|A_n|$  is divisible by 4. Moreover, if  $n=3k^2$  holds for some integer k, then the pairs (k,k),(-k,-k) belongs to  $A_n$ , and the remaining elements of  $A_n$  can be grouped into sets of four, showing that  $|A_n| \equiv 2 \pmod{4}$ . Similarly, if  $n=k^2$  holds for some integer k, then the pairs (k,-k),(-k,k) belongs to  $A_n$ , and the remaining elements of  $A_n$  can be grouped into sets of four, showing that  $|A_n| \equiv 2 \pmod{4}$ . Thus,  $|A_n|$  is divisible by 4 if and only if n/3 is not a perfect square and n is not a perfect square.

#### References

[Che25] EVAN CHEN. The OTIS Excerpts. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)