# **Quadratic polynomials**

## MOPSS

22 June 2024



### Suggested readings

- Evan Chen's
  - advice On reading solutions, available at https://blog.evanchen. cc/2017/03/06/on-reading-solutions/.
  - Advice for writing proofs/Remarks on English, available at https: //web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

#### List of problems and examples

1.1	Example (Hungary MO 2001/02, Grades 11 and 12 — tech-
	nical schools, P2) $\ldots \ldots \ldots \ldots \ldots \ldots \ldots 2$
1.2	Example (Canada CMO 1971 P4) 3
1.3	Example (India RMO 2003 P6) 3
1.4	Example (All-Russian MO 2007 Grade 8 P1)
1.5	Example (India RMO 2007 P3) 5
1.6	Example (India RMO 2010 P2) 6
1.7	Example (India RMO 2012f P1) 6
1.8	Example (India RMO 2015d P2)
1.9	Example (India RMO 2015a P2)
1.10	Example (India RMO 2015b P2)
1.11	Example (India RMO 2015e P2)

#### §1 Quadratic polynomials

**Example 1.1** (Hungary MO 2001/02, Grades 11 and 12 — technical schools, P2). Consider the following 2000 equations:

$$1x^{2} + 2 \cdot 2x + 3 = 0$$
  

$$2x^{2} + 2 \cdot 3x + 4 = 0$$
  

$$3x^{2} + 2 \cdot 4x + 5 = 0$$
  

$$\vdots$$
  

$$2000x^{2} + 2 \cdot 2001x + 2002 = 0.$$

For each equation, consider the product of the sum of the real roots and the sum of their reciprocals (if it exists). What is the product of these products?

**Solution 1.** Note that a quadratic polynomial with nonzero constant term has nonzero roots, and hence the reciprocals of its roots exist. Moreover, if  $\alpha, \beta$  denote the roots of a quadratic polynomial  $ax^2 + bx + c$  with  $ac \neq 0$ , then  $1/\alpha, 1/\beta$  are the roots of quadratic polynomial  $cx^2 + bx + a$ . Note that

$$\alpha+\beta=-\frac{b}{a},\frac{1}{\alpha}+\frac{1}{\beta}=-\frac{b}{c}$$

Since

$$(2(n+1))^2 - 4n(n+2) = 4n^2 + 8n + 4 - 4n^2 - 8n = 4 \ge 0,$$

it follows that the given 2000 equations have real roots. The product of the products considered is equal to

$$\frac{2^2 \cdot 2^2}{1 \cdot 3} \times \frac{2^2 \cdot 3^2}{2 \cdot 4} \times \frac{2^2 \cdot 4^2}{3 \cdot 5} \times \frac{2^2 \cdot 5^2}{4 \cdot 6} \times \dots \times \frac{2^2 \cdot 2000^2}{1999 \cdot 2001} \times \frac{2^2 \cdot 2001^2}{2000 \cdot 2002}$$

$$= \frac{4^{2000}}{1 \cdot 2} \frac{2^2 \cdot 3^2 \cdot \dots \cdot 2001^2}{3^2 \cdot 4^2 \cdot \dots \cdot 2000^2} \frac{1}{2001 \cdot 2002}$$
  
= 
$$\frac{4^{2000} \cdot 2^2 \cdot 2001^2}{1 \cdot 2 \cdot 2001 \cdot 2002}$$
  
= 
$$\frac{2001}{1001} \times 2^{4000}.$$

**Example 1.2** (Canada CMO 1971 P4). Determine all real numbers a such that the two polynomials  $x^2 + ax + 1$  and  $x^2 + x + a$  have at least one root in common.

**Solution 2.** Let *a* be a real nuber such that the two polynomials  $x^2 + ax + 1$  and  $x^2 + x + a$  have at least one root in common. Let  $\alpha$  denote a common root of these polynomials. The equations

$$\alpha^2 + a\alpha + 1 = 0, \alpha^2 + \alpha + a = 0$$

yield

$$(a-1)\alpha = a-1.$$

If  $a \neq 1$ , then  $\alpha = 1$  and hence a = -2. This proves that a = 1, -2.

If a = 1, then the given polynomials have at least one root in common. If a = -2, then the given polynomials vanish at 1.

We conclude that a = 1, -2 are precisely all the real numbers such that the given polynomials have at least one common root.

**Example 1.3** (India RMO 2003 P6). Find all real numbers a for which the equation

$$x^2 + (a-2)x + 1 = 3|x|$$

has exactly three distinct real solutions for x.

**Solution 3.** Let a be a real number such that  $x^2 + (a-2)x + 1 = 3|x|$  has exactly three distinct real solutions. Note that the equation

$$(x^{2} + (a - 2)x + 1 - 3x)(x^{2} + (a - 2)x + 1 + 3x) = 0$$

also has exactly three distinct real solutions. It follows that the discriminant of one of the polynomials  $x^2 + (a - 2)x + 1 - 3x$ ,  $x^2 + (a - 2)x + 1 + 3x$  vanishes, and the discriminant of the other is positive. The discriminants of these polynomials are

$$(a-5)^2 - 4 = a^2 - 10a + 21$$
  
=  $(a-3)(a-7)$ ,

$$(a+1)^2 - 4 = a^2 + 2a - 3$$
  
=  $(a+3)(a-1)$ 

respectively. It follows that a does not belong to  $(-\infty, -3) \cup (-3, 1) \cup (1, 3) \cup (3, 7) \cup (7, \infty)$ , or equivalently, a belongs to  $\{-3, 1, 3, 7\}$ .

Let us consider the case that a lies in  $\{-3, 1, 3, 7\}$ . Note that then one of (a-3)(a-7), (a+3)(a-1) vanishes and another is positive, and consequently, one of the polynomials  $x^2 + (a-2)x + 1 - 3x$ ,  $x^2 + (a-2)x + 1 + 3x$  have distinct real roots, and the other has a double root, which is a real number. Observe that

$$\left(\frac{a-5}{2}\right)^2 - (a+1)\left(\frac{a-5}{2}\right) + 1 = \frac{1}{4}\left(a^2 - 10a + 25 - 2a^2 + 8a + 10 + 4\right)$$
$$= \frac{1}{4}\left(-a^2 - 2a + 39\right),$$

which has a negative discriminant. This shows that if  $x^2 + (a-2)x + 1 - 3x$  has a real double root, then that cannot be a root of the other polynomial. Also note that

$$\left(\frac{a+1}{2}\right)^2 - (a-5)\frac{a+1}{2} + 1 = \frac{1}{4}\left(a^2 + 2a + 1 - 2a^2 + 8a + 10 + 4\right)$$
$$= \frac{1}{4}\left(-a^2 + 10a + 15\right),$$

whose roots are not integers. Using that a is an integer, it follows that if  $x^2 + (a-2)x + 1 + 3x$  has a real double root, then that cannot be a root of  $x^2 + (a-2)x + 1 - 3x$ . We conclude that if a lies in  $\{-3, 1, 3, 7\}$ , then

$$(x^{2} + (a - 2)x + 1 - 3x)(x^{2} + (a - 2)x + 1 + 3x) = 0$$

has exactly three distinct real solutions, or equivalently, the equation

$$x^2 + (a-2)x + 1 = 3|x|$$

has exactly three distinct real solutions for x.

So the required real numbers are a = -3, 1, 3, 7.

**Example 1.4** (All-Russian MO 2007 Grade 8 P1). If a, b, c are real numbers, show that at least one of the equations

$$x^{2} + (a - b)x + (b - c) = 0,$$
  

$$x^{2} + (b - c)x + (c - a) = 0,$$
  

$$x^{2} + (c - a)x + (a - b) = 0$$

has a real solution.

**Solution 4.** The sum of the discriminants of the above quadratic polynomials is

$$(a-b)^2 - 4(b-c) + (b-c)^2 - 4(c-a) + (c-a)^2 - 4(a-b)$$
  
=  $(a-b)^2 + (b-c)^2 + (c-a)^2$ ,

which is positive if not all of a, b, c are equal. Consequently, if not all of the three real numbers a, b, c are equal, then at least one of the quadratic polynomials

$$x^{2} + (a - b)x + (b - c), x^{2} + (b - c)x + (c - a), x^{2} + (c - a)x + (a - b)$$

has positive discriminant, and hence admits real solutions. Moreover, if all of a, b, c are equal, at least one (in fact, all) of the above polynomials admits a real root.

**Example 1.5** (India RMO 2007 P3). Find all pairs (a, b) of real numbers such that whenever  $\alpha$  is a root of  $x^2 + ax + b = 0$ ,  $\alpha^2 - 2$  is also a root of the equation.

**Solution 5.** Let a, b be real numbers such that for any root  $\alpha$  of  $x^2 + ax + b = 0$ ,  $\alpha^2 - 2$  is also a root. Denote the roots of  $x^2 + ax + b$  by  $\alpha, \beta$ . There are the following possibilities.

- (1)  $\alpha^2 2 = \alpha, \beta^2 2 = \beta,$
- (2)  $\alpha^2 2 = \beta, \ \beta^2 2 = \alpha,$
- (3)  $\alpha^2 2 = \beta^2 2 = \alpha$ ,
- (4)  $\alpha^2 2 = \beta^2 2 = \beta$ .

If  $\alpha = \beta$ , then these four cases are equivalent to

$$\alpha^2 - 2 = \beta^2 - 2 = \alpha = \beta,$$

which shows that  $\alpha$  is equal to 2 or -1, and hence (a, b) is equal to (-4, 4) or (2, 1).

It remains to consider the case that  $\alpha \neq \beta$ , which we assume from now on.

In Case (1),  $\alpha, \beta$  satisfy the equation  $X^2 - X - 2 = 0$ . So  $(\alpha, \beta)$  is equal to (2, -1) or (-1, 2), and hence (a, b) is equal to (-1, -2).

In Case (2), we have  $\alpha^2 - \beta^2 = \beta - \alpha$ , which gives  $\alpha + \beta = -1$  (since  $\alpha \neq \beta$ ). So

$$\alpha\beta = \frac{1}{2}(\alpha+\beta)^2 - \frac{1}{2}(\alpha^2+\beta^2) = \frac{1}{2}(\alpha+\beta)^2 - \frac{1}{2}(\alpha+\beta+4) = \frac{1}{2} - \frac{3}{2} = -1.$$

This shows that  $\alpha, \beta$  are roots of the quadratic polynomial  $x^2 + x - 1$ , and hence, (a, b) is equal to (1, -1).

In Case (3), note that  $\alpha$  is equal to 2 or -1. Using  $\beta^2 = 2 + \alpha$  and  $\alpha \neq \beta$ , it follows that  $(\alpha, \beta)$  is equal to (2, -2) or (-1, 1), and hence (a, b) is equal to (0, -4) or (0, -1).

Similarly, in Case (4),  $(\alpha, \beta)$  is equal to (-2, 2) or (-1, 1), which shows (a, b) is equal to (0, -4) or (0, -1).

So (a, b) is equal to one of (-4, 4), (2, 1), (-1, -2), (1, -1), (0, -4), (0, -1).

Moreover, if (a, b) is equal to any of these six pairs, then it can be checked that for any root  $\alpha$  of  $x^2 + ax + b = 0$ ,  $\alpha^2 - 2$  is also a root.

We conclude that all the required pairs are (-4, 4), (2, 1), (-1, -2), (1, -1), (0, -4), (0, -1).

Example 1.6 (India RMO 2010 P2). Let

$$P_1(x) = ax^2 - bx - c, P_2(x) = bx^2 - cx - a, P_3(x) = cx^2 - ax - b$$

be three quadratic polynomials where a, b, c are nonzero real numbers. Suppose there exists a real number  $\alpha$  such that  $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$ . Prove that a = b = c.

**Solution 6.** Since  $P_1(\alpha), P_2(\alpha)$  are equal, we get

$$(a-b)\alpha^2 = (b-c)\alpha + (c-a),$$

which gives

$$(a-b)(\alpha^{2}+1) = (b-c)(\alpha-1).$$

Similarly, using  $P_2(\alpha) = P_3(\alpha)$ , we obtain

$$(b-c)(\alpha^2+1) = (c-a)(\alpha-1).$$

If  $\alpha = 1$ , then it follows that a = b = c. Henceforth, let us assume that  $\alpha \neq 1$ . Then the above yields

$$(a-b)(c-a) = (b-c)^2.$$

Using a similar argument as above, it follows that

$$(b-c)(a-b) = (c-a)^2, (c-a)(b-c) = (a-b)^2.$$

Adding these equations, we obtain

$$a^2 + b^2 + c^2 = ab + bc + ca.$$

Since a, b, c are real, it follows that a = b = c. This completes the proof.

**Example 1.7** (India RMO 2012f P1). Find nonzero real numbers a, b such that  $x^2 + ax + b, x^2 + x + ab, ax^2 + x + b$  are three distinct polynomials with a common root.

**Solution 7.** Let a, b be real numbers such that  $x^2 + ax + b, x^2 + x + ab, ax^2 + x + b$  are three distinct polynomials with a common root  $\alpha \in \mathbb{C}$ . We obtain

$$\alpha^2+a\alpha+b=\alpha^2+\alpha+ab=a\alpha^2+\alpha+b=0,$$

which gives  $a\alpha + b = \alpha + ab$ , that is,  $(a-1)(\alpha - b) = 0$ . Since  $x^2 + ax + b$ ,  $x^2 + x + ab$ ,  $ax^2 + x + b$  are distinct, it follows that  $a \neq 1$ . This shows that  $\alpha = b$ . Since the polynomials  $x^2 + x + ab$ ,  $ax^2 + x + b$  vanish at  $x = \alpha = b$ , we obtain

$$b(a+b+1) = b(ab+2) = 0.$$

Using b is nonzero, we get a + b + 1 = ab + 2 = 0. Note that 1 + ab - a - b = 0. Since  $a \neq 1$ , we obtain b = 1, which combined with ab + 2 = 0 implies that a = -2.

Also note that for a = -2, b = 1, the given polynomials are equal to

$$x^{2} - 2x + 1, x^{2} + x - 2, -2x^{2} + x + 1,$$

which are all distinct and they vanish at x = 1.

We conclude that precisely for (a, b) = (-2, 1), the given polynomials are all distinct and have a common root.

**Example 1.8** (India RMO 2015d P2). Let  $P(x) = x^2 + ax + b$  be a quadratic polynomial with real coefficients. Suppose there are real numbers s is not equal to t such that P(s) = t and P(t) = s. Prove that b - st is a root of the equation  $x^2 + ax + b - st = 0$ .

Solution 8. We have

$$s^{2} + as + b = t, t^{2} + at + b = s.$$

Taking their difference, we obtain (s - t)(s + t + a + 1) = 0, which gives s + t + a + 1 = 0 since  $s \neq t$ . Using the above, we obtain

$$s(s^{2} + as + b) - t(t^{2} + at + b) = 0,$$

or equivalently,

$$(s-t)(b+a(s+t)+s^2+st+t^2) = 0.$$

Combining the above with s + t + a + 1 = 0 and  $s \neq t$ , we obtain

$$b - (s+t) - st = 0.$$

Note that

$$(b-st)^2 + a(b-st) + b - st = (b-st)(b-st+a+1)$$
  
=  $(b-st)(s+t+a+1)$   
= 0.

This completes the proof.

**Example 1.9** (India RMO 2015a P2). Let  $P(x) = x^2 + ax + b$  be a quadratic polynomial where a, b are real numbers. Suppose  $\langle P(-1)^2, P(0)^2, P(1)^2 \rangle$  is an AP of positive integers. Prove that a, b are integers.

**Solution 9.** Note that  $P(-1)^2$ ,  $P(0)^2$ ,  $P(1)^2$  are equal to

$$1 + a^2 + b^2 - 2a + 2b - 2ab, b^2, 1 + a^2 + b^2 + 2a + 2b + 2ab$$

respectively. Since they form an arithmetic progression, we obtain

$$1 + a^{2} + b^{2} - 2a + 2b - 2ab + 1 + a^{2} + b^{2} + 2a + 2b + 2ab = 2b^{2},$$

or equivalently,  $a^2 + 2b + 1 = 0$ . It follows that

$$b^2 - 2a - 2ab, b^2, b^2 + 2a + 2ab$$

form an arithmetic progression of positive integers. Note that

$$(2a + 2ab)^{2} = 4a^{2}(b+1)^{2}$$
  
= -4(2b+1)(b+1)^{2}  
= -4(2b^{3} + 5b^{2} + 4b + 1)  
= -4((2b^{2} + 4)b + 5b^{2} + 1).

Since  $b^2$  is an integer, it follows that b is rational number. Since b is rational and  $b^2$  is an integer, it follows that b is an integer. Using  $a^2 + 2b + 1 = 0$ , it follows that  $a^2$  is an integer. Moreover, if b = -1, then a is an integer. If  $b \neq -1$ , then using that 2a + 2ab is an integer, we obtain a is rational. Since  $a^2$  is an integer and a is rational, it follows that a is an integer. This completes the proof.

**Example 1.10** (India RMO 2015b P2). Let  $P(x) = x^2 + ax + b$  be a quadratic polynomial where *a* is real and  $b \neq 2$ , is rational. Suppose  $P(0)^2$ ,  $P(1)^2$ ,  $P(2)^2$  are integers, prove that *a* and *b* are integers.

**Solution 10.** Since b is rational and  $P(0)^2 = b^2$  is an integer, it follows that b is an integer. Note that

$$P(1)^{2} = (1 + a + b)^{2}$$
  
= 1 + a<sup>2</sup> + b<sup>2</sup> + 2a + 2b + 2ab,  
$$P(2)^{2} = (4 + 2a + b)^{2}$$
  
= 16 + 4a<sup>2</sup> + b<sup>2</sup> + 16a + 8b + 4ab.

Since b is an integer, the given conditions imply that  $a^2+2a+2ab$ ,  $4a^2+16a+4ab$  are integers. This shows that

$$4a^{2} + 16a + 4ab - 2(a^{2} + 2a + 2ab) = 2a^{2} + 12a,$$

$$4a^{2} + 16a + 4ab - 4(a^{2} + 2a + 2ab) = 8a - 4ab$$

are integers. Since  $b \neq 2$  and b is an integer, it follows that a is a rational number. Combining this with the fact that  $2a^2 + 12a$  is rational, it follows that a is equal to  $\frac{n}{2}$  for some integer n. Indeed, write  $a = \frac{x}{y}$  where x, y are integers with  $y \geq 1$  and gcd(x, y) = 1. Note that  $2\frac{x^2}{y} + 12x$  is an integer. Since x and y are relatively prime, this implies that y divides 2. Consequently, a is equal to  $\frac{n}{2}$  for some integer n. Using that  $2a^2 + 12a$  is an integer, we get that  $\frac{n^2}{2}$  is also an integer. This shows that n is even, and hence a is an integer. This completes the proof.

**Example 1.11** (India RMO 2015e P2). Let  $P_1(x) = x^2 + a_1x + b_1$  and  $P_2(x) = x^2 + a_2x + b_2$  be two quadratic polynomials with integer coefficients. Suppose  $a_1 \neq a_2$  and there exist integers  $m \neq n$  such that  $P_1(m) = P_2(n), P_2(m) = P_1(n)$ . Prove that  $a_1 - a_2$  is even.

**Solution 11.** Using  $P_1(m) = P_2(n)$ , we get

$$m^2 + a_1m + b_1 = n^2 + a_2n + b_2,$$

that is,

$$(m2 - n2) + (a_1m - a_2n) + b_1 - b_2 = 0.$$

Similarly, using  $P_1(n) = P_2(m)$ , we get

$$(n2 - m2) + (a_1n - a_2m) + b_1 - b_2 = 0.$$

This yields

$$2(m^{2} - n^{2}) + (a_{1} + a_{2})(m - n) = 0.$$

Since  $m \neq n$ , we get  $2(m+n) + a_1 + a_2 = 0$ . It follows that  $a_1 + a_2$  is even, and hence, so is  $a_1 + a_2 - 2a_2 = a_1 - a_2$ .