An Introduction to Mathematical Olympiads

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Outline of the talk

IMO

- The International Mathematical Olympiad (IMO)
  - World Championship Mathematics Competition for High School students.
  - Held annually in a different country.
- First IMO — Romania, 1959 — 7 countries participated.
- Currently, over 100 countries from 5 continents participate.
- Each country sends a team of up to six students.
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  - IMO2023 — Japan.
  - IMO2022 — Norway.
  - IMO2021 — Russian Federation.
  - IMO2020 — Russian Federation.

IMO cont ...

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  - IMO2024 — UK.
  - IMO2025 — Australia.
  - IMO2026 — People's Republic of China.
  - IMO2027 — Hungary.
- IMO held over two days.
  - Q1, Q2, Q3 on Day 1 — 4.5 hours.
  - Q4, Q5, Q6 on Day 2 — 4.5 hours.
  - Q1 $\leq$ Q4 $\leq$ Q2 $\leq$ Q5 $\leq$ Q3 $\leq$ Q6.

Participation of India in IMO

- India has been participating in IMO since 1989, and has been a host in 1996.
- India has received
  - 16 Gold medals,
  - 73 Silver medals,
  - 79 Bronze medals,
  - 28 Honourable mentions.
- India ranked
  - 7th in 1998 (G, G, G, S, S, S),
  - 7th in 2001 (G, G, S, S, B, B),
  - 9th in 2002 (G, S, S, S, B, B),
  - 11th in 2012 (G, G, S, S, S, HM),
  - 9th in 2023 (G, G, S, S, B, B).
- At least one Gold medal in each IMO held during 2019 - 2023.

cont ...

Some of the participants of the recent IMOs

- Anant Mudgal
  - IMO2015 (HM), IMO2016 (B), IMO2017 (B), IMO2018 (S)
  - An alumni of the Chennai Mathematical Society
  - A student at the University of California San Diego
- Pranjal Srivastava is the first participant from India who received **three Gold medals** in IMO (2019, 2021, 2022).
  - IMO2018 (S)
  - IMO's Hall of Fame
  - Bronze medal in IOI 2021 (International Olympiad in Informatics)
  - A student at MIT
- Atul Shatavart Nadig
  - IMO2022 (B), IMO2023 (G)
  - A student at MIT

cont ...

- Some of the past contestants are
  - Chetan Balwe, IISER Mohali
  - Riddhipratim Basu, ICTS
  - Ashay Burungale, University of Texas at Austin
  - Swarnendu Datta, IISER Kolkata
  - Subhash Khot, New York University, received Rolf Nevanlinna Prize 2014, a Fellow of the Royal Society
  - Abhinav Kumar, a mathematician working in industry
  - Kartik Prasanna, University of Michigan, Ann Arbor
  - Abhishek Saha, Queen Mary University of London
  - Sucharit Sarkar, University of California at Los Angeles
  - Kannan Soundararajan, Stanford University
  - Vaibhav Vaish, IISER Mohali

Participation of India in Mathematical Olympiads

- International Mathematical Olympiad (IMO) since 1989.
- Asian Pacific Mathematics Olympiad (APMO) since 2015.
- European Girls' Mathematical Olympiad (EGMO) since 2015.

Mathematical Olympiad program in India

- Organized by the Homi Bhabha Centre for Science Education (HBCSE), on behalf of the National Board for Higher Mathematics (NBHM).
- Eligibility and its stages
  - Students enrolled in the 8th, 9th, 10th, 11th or 12th standard may participate, provided certain additional conditions are met. The precise details are available at the webpage of HBCSE.
  - Indian Olympiad Qualifier in Mathematics (IOQM).
  - Regional Mathematical Olympiad (RMO).
  - Indian National Mathematical Olympiad (INMO).
  - International Mathematical Olympiad Training Camp (IMOTC). Through TSTs, leads to the selection of a team of six students to represent India at IMO.
  - Pre-Departure Camp (PDC) held before leaving for IMO.
- The Math Olympiad program organized by HBCSE, is the **only one** leading to participation in the International Mathematical Olympiads. **No other contests are recognized**.

From Madhya Pradesh

- 144 students qualified in IOQM 2023
- 39 students qualified in RMO 2023

Syllabus

- Algebra
- Combinatorics
- Geometry
- Number Theory

A1

Problem

Show that the positive integers of the form $4n+3$, that is, the integers

$$3, 7, 11, 15, 19, \ldots$$

cannot be written as the sum of two perfect squares.

Walkthrough.

(a) Consider the integers

$$\begin{aligned}
&0^2+1^2, 0^2+2^2, 0^2+3^2, 0^2+4^2, \ldots, \\
&1^2+1^2, 1^2+2^2, 1^2+3^2, 1^2+4^2, \ldots, \\
&2^2+1^2, 2^2+2^2, 2^2+3^2, 2^2+4^2, \ldots, \\
&3^2+1^2, 3^2+2^2, 3^2+3^2, 3^2+4^2, \ldots, \\
&4^2+1^2, 4^2+2^2, 4^2+3^2, 4^2+4^2, \ldots.
\end{aligned}$$

A1 cont. . .

(b) Observe that upon division by 4, they leave the integers $0, 1, 2$ as remainders.

$$
\begin{aligned}
&0^2+1^2 \rightsquigarrow \mathbf{1}, 0^2+2^2 \rightsquigarrow \mathbf{0}, 0^2+3^2 \rightsquigarrow \mathbf{1}, 0^2+4^2 \rightsquigarrow \mathbf{0}, \dots, \\
&1^2+1^2 \rightsquigarrow \mathbf{2}, 1^2+2^2 \rightsquigarrow \mathbf{1}, 1^2+3^2 \rightsquigarrow \mathbf{2}, 1^2+4^2 \rightsquigarrow \mathbf{1}, \dots, \\
&2^2+1^2 \rightsquigarrow \mathbf{1}, 2^2+2^2 \rightsquigarrow \mathbf{0}, 2^2+3^2 \rightsquigarrow \mathbf{1}, 2^2+4^2 \rightsquigarrow \mathbf{0}, \dots, \\
&3^2+1^2 \rightsquigarrow \mathbf{2}, 3^2+2^2 \rightsquigarrow \mathbf{1}, 3^2+3^2 \rightsquigarrow \mathbf{2}, 3^2+4^2 \rightsquigarrow \mathbf{1}, \dots, \\
&4^2+1^2 \rightsquigarrow \mathbf{1}, 4^2+2^2 \rightsquigarrow \mathbf{0}, 4^2+3^2 \rightsquigarrow \mathbf{1}, 4^2+4^2 \rightsquigarrow \mathbf{0}, \dots.
\end{aligned}
$$

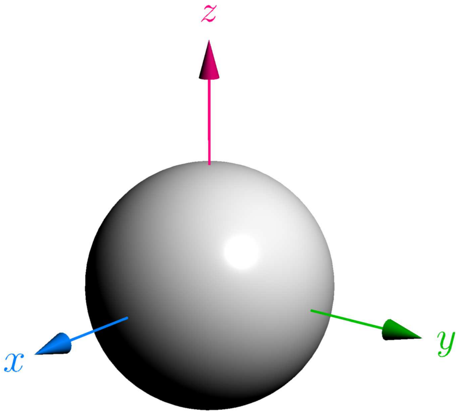
(c) Show that it is **always** the case, namely, upon division by 4, the sum of two perfect squares leaves one of $0, 1, 2$ as the remainder.

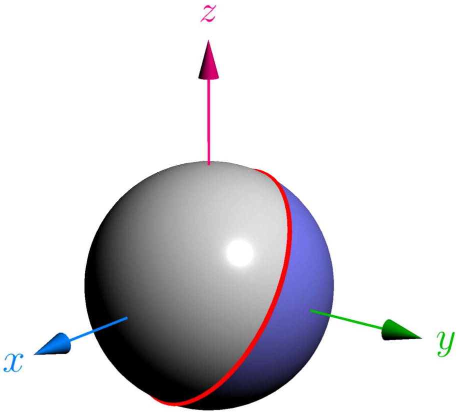
(d) Conclude that no integer, which leaves the remainder of 3 **upon division by** 4, can be written as the sum of two squares.

C1

Problem (Putnam 2002 A2)

Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.





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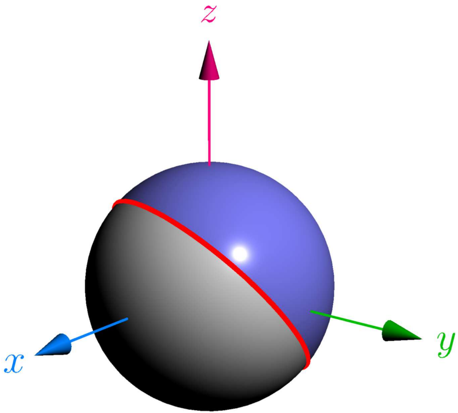
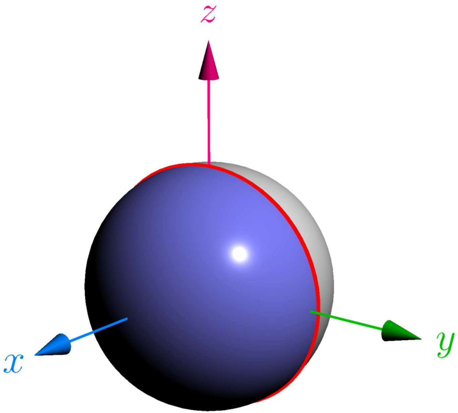




Walkthrough.

(a) Draw a great circle passing through at least two of the five points.

(b) At least one closed hemisphere contains at least two of the remaining three points.

(c) Conclude!

C2

Problem (Moscow MO 2015 Grade 11)

Prove that it is impossible to put the integers from 1 *to* 64 *(using each integer once) into an* 8×8 *table so that any* 2×2 *square, considered as a matrix, has a determinant that is equal to* 1 *or* -1.

- Given a 2×2 matrix (or array of numbers) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,
- its determinant is $ad - bc$, i.e.,

the product of the diagonal terms—the product of the anti-diagonal terms.

- $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ has determinant -2.
- $\begin{pmatrix} 8 & 9 \\ 7 & 12 \end{pmatrix}$ has determinant $96 - 63 = 33$.
- $\begin{pmatrix} 13 & 14 \\ 5 & 7 \end{pmatrix}$ has determinant $91 - 70 = 21$.

C2 cont ...

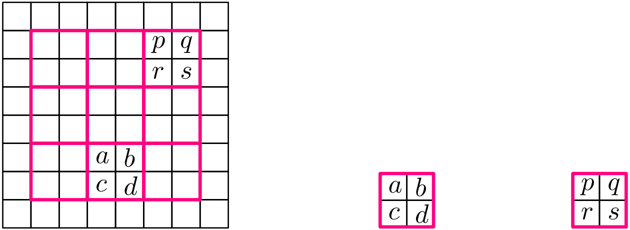


Figure: $ad - bc = \pm 1, ps - qr = \pm 1$

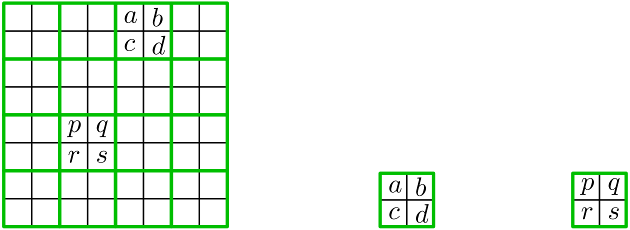
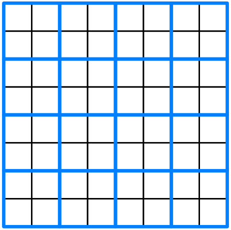


Figure: $ad - bc = \pm 1, ps - qr = \pm 1$

C2 cont . . .

Walkthrough.



(a) Assume that such a filling exists.

(b) Recall that the determinant of a 2×2 square $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$ is

the product of the diagonal terms−the product of the anti-diagonal terms.

(c) Note that $\boxed{\text{even} - \text{even} \neq \pm 1, \text{odd} - \text{odd} \neq \pm 1}$, and hence any square contains two odd numbers along the diagonal or on the anti-diagonal.

(d) Divide the 8×8 table into 16 pairwise disjoint 2×2 squares.

(e) Each of these 16 squares contains at least two odd integers, and hence, they together contain at least 32 odd integers.

(f) Conclude that each of these 16 squares contains precisely two odd integers, and precisely two even integers.

C2 cont . . .

(g) Consider a square among them. It is of the form

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array}$$ with a, d both odd, and b, c both even,

or of the form

$$\begin{array}{|c|c|}\hline b & a \\ \hline d & c \\ \hline \end{array}$$ with a, d both odd, and b, c both even.

(h) The product of its even entries is at most one more than the product of its odd entries.

(i) Note that for any two odd positive integers b, c, the inequality $bc + 1 < (b + 1)(c + 1)$ holds.

(j) This shows that

the product of two evens between 1 and 64
$<$ the product of two (possibly different) evens between 1 and 64.

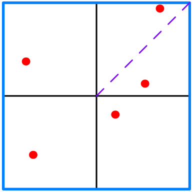
(k) Multiply all the even entries of the 16 squares to obtain

$$2 \cdot 4 \cdot \ldots \cdot 64 < (1 + 1) \cdot (3 + 1) \cdot \ldots \cdot (63 + 1) = 2 \cdot 4 \cdot \ldots \cdot 64.$$

G1

Problem

Among any 5 points in a 2×2 square, show that there are two points which are at most $\sqrt{2}$ apart.



Walkthrough.

(a) Divide the 2×2 square into four unit squares.

(b) Two points among any choice of 5 points from the 2×2 square lie in one of these unit squares.

(c) The distance between any two points lying in a unit square is at most the length of any of its diagonals, that is, at most $\sqrt{2}$.

N1

Problem (Austrian Junior Regional Competition 2022)

Determine all prime numbers p, q and r with $p + q^2 = r^4$.

Walkthrough.

(a) Note that

$$\begin{aligned} p &= r^4 - q^2 \\ &= (r^2 - q)(r^2 + q). \end{aligned}$$

(b) This gives $r^2 - q = 1$, and hence

$$\begin{aligned} q &= r^2 - 1 \\ &= (r - 1)(r + 1). \end{aligned}$$

(c) This implies that $r - 1 = 1$.

(d) Conclude that $r = 2, q = 3, p = 7$.

N2

Problem (cf. Australian Mathematics Competition 1984)

Suppose

$$x_1, x_2, x_3, x_4, \ldots$$

is a sequence of integers satisfying the following properties:

(1) $x_2 = 2$,

(2) $x_{mn} = x_m x_n$ *for all positive integers* m, n,

(3) $x_m < x_n$ *for any positive integers* m, n *with* $m < n$.

Find x_{2024}.

Walkthrough.

(a) What can be said about x_4, x_8, x_{16}, x_{32}?

(b) Note that $x_4 = x_{2\times 2}, x_8 = x_{4\times 2}, x_{16} = x_{8\times 2}, x_{32} = x_{16\times 2}$.

(c) Can one show that $x_{2^n} = 2^n$ for any $n \geq 1$?

(d) Show that $x_m = m$ for any $m \geq 1$ (does property (3) help?).

Preparation

- Pick up any standard textbook to work through, so you learn some of the standard theory that is tested in math contests.
- Go through some past problems from previous contests.
- Rope some friends into learning with you. It's more fun that way, and you can learn from each other.
- You should repeat these steps until you have some comfort with the kinds of problems that appear.
- As you get experience, you will automatically start to know what deep understanding feels like.
- Be aware that you will see many, many problems which you can't solve, where you read the solution and ask, "how was I supposed to think of that?". This is okay and expected: it's not because you're dumb, it's because you are learning.

These are some of the suggestions from Evan Chen.

Evan Chen

Evan Chen is a graduate student at MIT and a math olympiad coach. He received a Gold medal in IMO 2014.

- FAQs about math contests and particularly how to go about training for them
- Olympiad Articles
  - Math olympiad beginner's page
  - Math olympiad coach's page
- Recommended Readings (includes handouts by Evan Chen, Yufei Zhao, Po-Shen Loh, Alex Remorov, and suggests references)

OTIS by Evan Chen

- Evan Chen runs the Olympiad Training for Individual Study (OTIS), a proof-based olympiad training program, with over 300 students per year from across the world. Some of its alumnis are
  - Anant Mudgal, participated at IMO in 2015 (HM), 2016 (B), 2017 (B), 2018 (S).
  - Pranjal Srivastava, participated at IMO in 2018 (S), 2019 (G), 2021 (G), 2022 (G).
  - Atul Shatavart Nadig, participated in IMO in 2022 (B), 2023 (G).
  - Anushka Aggarwal, received bronze medals in EGMO in 2019, 2020, 2022.

References

- Challenge and Thrill of Pre-College Mathematics by V. Krishnamurthy, C.R. Pranesachar, K.N. Ranganathan, B.J. Venkatachala
- Olympiad Combinatorics, by Pranav A. Sriram, is an intermediate-advanced textbook.
- Euclidean Geometry in Mathematical Olympiads (EGMO) by Evan Chen.
- OTIS Excerpts by Evan Chen for non-geometry.
- Olympiad NT through Challenging Problems, by Justin Stevens, is an introductory olympiad text.
- Modern Olympiad Number Theory, by Aditya Khurmi, olympiad-oriented number theory textbook.
- Problems from the Book by Titu Andreescu and Gabriel Dospinescu. Intermediate-advanced textbook covering topics in inequalities, algebra, analysis, combinatorics, and number theory.

Resources

- On Art of Problem-Solving, there is an extensive archive of problems from basically every Math Competition, together with community-contributed solutions.

Problem Solving Sessions

- To be held in person.
- The aim is to develop an interest in mathematics among the students by encouraging them to work on problems falling broadly within the scope of Mathematical Olympiads.
- The first session will tentatively take place in August 2024.
- Applications to be accepted during June, 2024 through a Google form.
- A problem set will be available through the form. While filling in the form, the solutions to these problems (or the details of the progress made) are to be submitted.
- The students, selected for participation in the session, will be informed in July 15, 2024.
- For more information, you may write to
  - Kartick Adhikari (kartick@iiserb.ac.in), Office 307, AB1 (Academic Building 1),
  - Jyoti Prakash Saha (jpsaha@iiserb.ac.in), Office 206, AB1.
- Slides will be posted at `https://jpsaha.github.io/MOTP/`.

Thank you!