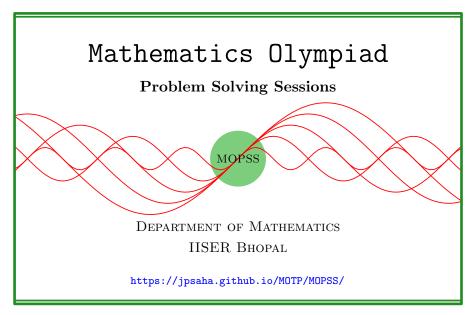
Quadratic polynomials

MOPSS

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Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog. evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [Che25, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https: //www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

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§1 Quadratic polynomials

Example 1.1 (India RMO 1991 P6). Find all integer values of a such that the quadratic expression

$$(x+a)(x+1991)+1$$

can be factored as a product (x+b)(x+c) where b and c are integers.

Solution 1. Let us establish the following claim.

Claim — A monic quadratic polynomial f(x) with integer coefficients factorizes into the product of two monic polynomials with integer coefficients if and only if the discriminant of f(x) is a perfect square.

Proof of the Claim. Since the discriminant of f(x) is equal to the square of the twice of the difference of its roots, the "only if part" follows.

To prove the "if part", assume that the discriminant of f(x) is a perfect square. Write $f(x) = x^2 + Bx + C$. Note that $B^2 - 4C = n^2$ for some integer n. This shows that the integers B, n are of the same parity. It follows that

$$-B+n, -B-n$$

are even integers. This implies that the polynomial f(x) have integer roots, proving the "if part" of the Claim.

By the above Claim, the given problem is equivalent to determining all the integers a such that the discriminant of

$$(x+a)(x+1991)+1$$

is a perfect square, that is,

$$(a+1991)^2 - 4(1991a+1) = n^2$$

holds for some integer n.

Let a, n be integers satisfying the above condition. Note that the above can be rewritten as

$$(a - 1991)^2 - n^2 = 4,$$

which is equivalent to

$$(a - 1991 - n)(a - 1991 + n) = 4.$$

Noting that the integers a - 1991 - n, a - 1991 + n are of the same parity, it follows that the above condition is equivalent to (a - 1991 - n, a - 1991 + n) being equal to one of (2, 2), (-2, -2), which holds if and only if (a, n) is equal to one of (1993, 0), (1989, 0).

This proves that the integer values of a satisfying the required condition are precisely 1989, 1993.

Example 1.2 (India RMO 1999 P7). Find the number of quadratic polynomials, $ax^2 + bx + c$, which satisfy the following conditions:

- 1. a, b, c are distinct,
- 2. $a, b, c \in \{1, 2, 3, \dots, 1999\}$ and
- 3. x + 1 divides $ax^2 + bx + c$.

Solution 2. Note the third condition is equivalent to a + c = b. So the given problem is equivalent to finding the number of triples (a, b, c) where a, b, c are distinct integers lying between 1 and 1999, and satisfy a + c = b. Note that it is equal to the number of pairs (a, c) of distinct integers a, c lying between 1 and 1999, such that their sum a + c is at most 1999. Observe that these are precisely the pairs of the form (a, c) where a, c are integers satisfying $1 \le a \le 1998$, $1 \le c \le 1999 - a$ and $a \ne c$. Note that the number of such pairs is equal to

$$\sum_{i=1}^{1998} (1999 - i) - 999,$$

where the sum accounts for the pairs (a, c) satisfying $1 \le a \le 1998$ and $1 \le c \le 1999 - a$, and the term 999 accounts for the pairs (a, c) satisfying these two conditions and the condition a = c. We conclude that the number of the quadratic polynomials, satisfying the given conditions, is equal to

$$\sum_{i=1}^{1998} (1999 - i) - 999 = \sum_{i=1}^{1998} i - 999$$

= 999 × 1999 - 999
= 1998000 - 1998
= 1996002.

Example 1.3 (India RMO 2013a P6). Suppose that m and n are integers such that both the quadratic equations $x^2 + mx - n = 0$ and $x^2 - mx + n = 0$ have integer roots. Prove that n is divisible by 6.

Solution 3. Since the given quadratic polynomials have integer roots, their discriminants are a perfect squares. It follows that there are nonnegative integers a, b satisfying $m^2 + 4n = a^2, m^2 - 4n = b^2$. This gives

$$2m^2 = a^2 + b^2, 8n = a^2 - b^2.$$

Note that if 3 divides m, then 3 divides a and b, and hence, 3 divides n. Also note that if $m \equiv \pm 1 \pmod{3}$ holds, then a^2, b^2 are congruent to 0 modulo 3, and hence, 3 divides n.

If m is an odd integer, then a is odd, and hence, m^2, a^2 are congruent to 1 modulo 8, which gives that 8 divides 4n, implying n is even. If m is even, then so are the integers a and b, and we have

$$2n = \left(\frac{a}{2}\right)^2 - \left(\frac{b}{2}\right)^2,$$

which implies that the integers a/2, b/2 are of the same parity, and hence, 4 divides $(a/2)^2 - (b/2)^2$, implying that n is even.

We conclude that n is divisible by 6.

References

[Che25] EVAN CHEN. The OTIS Excerpts. Available at https://web. evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)