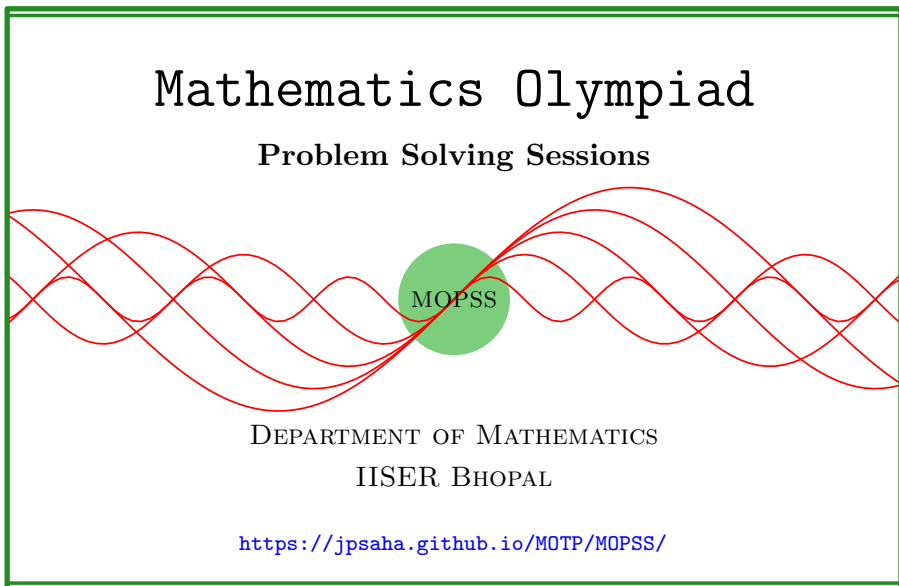


Induction

MOPSS

26 April 2025



Suggested readings

- Evan Chen's advice [On reading solutions](https://blog.evanchen.cc/2017/03/06/on-reading-solutions/), available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
- Evan Chen's [Advice for writing proofs/Remarks on English](https://web.evanchen.cc/handouts/english/english.pdf), available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- [Notes on proofs](#) by Evan Chen from [OTIS Excerpts](#) [Che25, Chapter 1].
- [Tips for writing up solutions](https://www.math.utoronto.ca/barbeau/writingup.pdf) by Edward Barbeau, available at <https://www.math.utoronto.ca/barbeau/writingup.pdf>.
- Evan Chen discusses why [math olympiads are a valuable experience for high schoolers](#) in the post on [Lessons from math olympiads](#), available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

List of problems and examples

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§1 Induction

Example 1.1 (India RMO 1993 P2). Prove that the ten's digit of any power of 3 is even.

Solution 1. For any positive integer n , note that

$$\begin{aligned}
 3^{2n} &= (10 - 1)^n \\
 &= \sum_{i=0}^n \binom{n}{i} 10^{n-i} (-1)^i \\
 &\equiv \binom{n}{n-1} 10(-1)^{n-1} + (-1)^n \pmod{100} \\
 &\equiv 10(-1)^{n-1}n + (-1)^n \pmod{100} \\
 &\equiv \begin{cases} (10 - n)10 + 1 \pmod{100} & \text{if } n \text{ is even,} \\ (n - 1)10 + 9 \pmod{100} & \text{if } n \text{ is odd.} \end{cases}
 \end{aligned}$$

Writing $n = 10k + r$ for some integers k, r with $1 \leq r \leq 10$, it follows that any power of 3 with a positive even exponent has an even number in its ten's digit.

For any positive integer n , observe that

$$3^{2n+1} \equiv \begin{cases} 3(10 - n)10 + 3 \pmod{100} & \text{if } n \text{ is even,} \\ (3(n - 1) + 2)10 + 7 \pmod{100} & \text{if } n \text{ is odd.} \end{cases}$$

Also note that the ten's digit of 3 is even. It follows that any power of 3 with a positive odd exponent has an even number in its ten's digit. ■

Here is a proof of Euclid's theorem on the infinitude of primes by Saidak.

Example 1.2 (Infinitude of primes). [Sai06] Let $a_1 = 2$ and $a_{n+1} = a_n(a_n + 1)$. Show that a_n has at least n distinct prime factors.

Walkthrough —

- (a) Show that $a_n \geq 2$ for any integer $n \geq 1$.
- (b) Note that the integers $a_n, a_n + 1$ have no common prime factor.
- (c) Conclude using induction.

References

- [Che25] EVAN CHEN. *The OTIS Excerpts*. Available at <https://web.evanchen.cc/excerpts.html>. 2025, pp. vi+289 (cited p. 1)
- [Sai06] FILIP SAIDAK. A new proof of Euclid’s theorem. In: *Amer. Math. Monthly*, **113**:10 (2006), pp. 937–938. ISSN: 0002-9890. DOI: [10.2307/27642094](https://doi.org/10.2307/27642094). URL: <http://dx.doi.org/10.2307/27642094> (cited p. 2)