Induction

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Induction

Example 1.1 (India RMO 1993 P2)**.** Prove that the ten's digit of any power of 3 is even.

Solution 1. For any positive integer n, note that

$$\begin{aligned}
3^{2n} &= (10-1)^n \\
&= \sum_{i=0}^{n} \binom{n}{i} 10^{n-i}(-1)^i \\
&\equiv \binom{n}{n-1} 10(-1)^{n-1} + (-1)^n \pmod{100} \\
&\equiv 10(-1)^{n-1} n + (-1)^n \pmod{100} \\
&\equiv \begin{cases} (10-n)10+1 \pmod{100} & \text{if } n \text{ is even,} \\ (n-1)10+9 \pmod{100} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Writing $n = 10k + r$ for some integers k, r with $1 \leq r \leq 10$, it follows that any power of 3 with a positive even exponent has an even number in its ten's digit.

For any positive integer n, observe that

$$3^{2n+1} \equiv \begin{cases} 3(10-n)10+3 \pmod{100} & \text{if } n \text{ is even,} \\ (3(n-1)+2)10+7 \pmod{100} & \text{if } n \text{ is odd.} \end{cases}$$

Also note that the ten's digit of 3 is even. It follows that any power of 3 with a positive odd exponent has an even number in its ten's digit. ■

Here is a proof of Euclid's theorem on the infinitude of primes by Saidak.

Example 1.2 (Infinitude of primes)**.** [**Sai06**] Let $a_1 = 2$ and $a_{n+1} = a_n(a_n+1)$. Show that a_n has at least n distinct prime factors.

> **Walkthrough** —
>
> **(a)** Show that $a_n \geq 2$ for any integer $n \geq 1$.
>
> **(b)** Note that the integers $a_n, a_n + 1$ have no common prime factor.
>
> **(c)** Conclude using induction.

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)

[**Sai06**] Filip Saidak. A new proof of Euclid's theorem. In: *Amer. Math. Monthly*, **113**:10 (2006), pp. 937–938. issn: 0002-9890. doi: 10.2307/27642094. url: http://dx.doi.org/10.2307/27642094 (cited p. 2)