Using identities

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics

IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Using identities

Example 1.1 (India RMO 2004 P3)**.** Let α and β be the roots of the quadratic equation $x^2 + mx - 1 = 0$, where m is an odd integer. Let $\lambda_n = \alpha^n + \beta^n$, for $n \geq 0$. Prove that for $n \geq 0$,

1. λ_n is an integer, and
2. $\gcd(\lambda_n, \lambda_{n+1}) = 1$.

Solution 1. For any positive integer n, note that

$$(\alpha + \beta)(\alpha^n + \beta^n) = \alpha^{n+1} + \beta^{n+1} + \alpha\beta(\alpha^{n-1} + \beta^{n-1}),$$

which implies that

$$\begin{aligned} \alpha^{n+1} + \beta^{n+1} &= (\alpha + \beta)(\alpha^n + \beta^n) - \alpha\beta(\alpha^{n-1} + \beta^{n-1}) \\ &= -m(\alpha^n + \beta^n) + (\alpha^{n-1} + \beta^{n-1}), \end{aligned}$$

which yields that

$$\lambda_{n+1} = -m\lambda_n + \lambda_{n-1}. \tag{1}$$

For any integer $n \geq 1$, let $P(n)$ denote the statement that $\lambda_0, \lambda_1, \ldots, \lambda_n$ are integers, and $\gcd(\lambda_{n-1}, \lambda_n) = 1$ holds. Note that $\lambda_0 = 2, \lambda_1 = -m$. Since m is an odd integer, it follows that $P(1)$ holds. Assume that the statement $P(k)$ holds for some positive integer k. Using Eq. (1), and using the induction hypothesis, it follows that λ_{k+1} is an integer. Using Eq. (1) once again, note that any common divisor of λ_k and λ_{k+1} divides λ_{k-1}, and hence is also a common divisor of λ_{k-1}, λ_k. Applying the induction, we conclude that the integers λ_k and λ_{k+1} are relatively prime. This proves that $P(k+1)$ holds. By the principle of induction, the statement $P(n)$ holds for any positive integer n. This completes the proof. ■

Example 1.2 (Flanders Mathematical Olympiad 2005 P3)**.** Prove that 2005^2 can be written in at least 4 ways as the sum of 2 perfect (nonzero) squares.

Solution 2. Note that $2005 = 5 \cdot 401$ holds, and $5, 401$ are primes. Observe that $5 = 1^2 + 2^2$ and $401 = 1^2 + 20^2$ hold. This gives

$$2005^2 = (5 \cdot 40)^2 + (5 \cdot 399)^2$$

$$= (3 \cdot 401)^2 + (4 \cdot 401)^2.$$

Using the identity

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2,$$

we obtain

$$5^2 = 3^2 + 4^2, \quad 401^2 = 399^2 + 40^2.$$

Using the above identity once again, it follows that

$$\begin{aligned} 2005^2 &= 5^2 \cdot 401^2 \\ &= (3^2 + 4^2)(40^2 + 399^2) \\ &= (3 \cdot 40 - 4 \cdot 399)^2 + (3 \cdot 399 + 4 \cdot 40)^2 \\ &= (3 \cdot 40 + 4 \cdot 399)^2 + (3 \cdot 399 - 4 \cdot 40)^2. \end{aligned}$$

This yields

$$\begin{aligned} 2005^2 &= (5 \cdot 40)^2 + (5 \cdot 399)^2 \\ &= (3 \cdot 401)^2 + (4 \cdot 401)^2 \\ &= (4 \cdot 399 - 3 \cdot 40)^2 + (3 \cdot 399 + 4 \cdot 40)^2 \\ &= (3 \cdot 40 + 4 \cdot 399)^2 + (3 \cdot 399 - 4 \cdot 40)^2. \end{aligned}$$

Note that the above four ways of expressing 2005^2 as a sum of two nonzero perfect squares are distinct[1]. ■

Example 1.3 (India RMO 2006 P6)**.** Prove that there are infinitely many positive integers n such that $n(n+1)$ can be expressed as a sum of two positive squares in at least two different ways. (Here $a^2 + b^2$ and $b^2 + a^2$ are considered as the same representation.)

Walkthrough —

(a) Note that if two integers can be expressed as sum of two squares, then their product can also be expressed as a sum of two squares.

(b) Also note that if n is a perfect square, then the product $n(n + 1)$ is a sum of two squares.

(c) If n is a perfect square, and it is a sum of two squares, that is, if $n = m^2$ and $n = a^2 + b^2$ hold, then

$$n(n + 1) = (m^2)^2 + m^2 = (am - b)^2 + (a + bm)^2.$$

(d) Note that the perfect squares which are sum of two (nonzero) squares correspond to the Pythagorean triples, which are precisely the triples

[1]How does it follow? Can one avoid computations to see this?

of the form, $(x^2 - y^2, 2xy, x^2 + y^2)$, where x, y are integers satisfying $x > y \geq 1$.

Solution 3. Let k be a positive integer. Note that

$$(k^2 - 1)^2 + (2k)^2 = (k^2 + 1)^2$$

holds. Put $n = (k^2 + 1)^2$. Observe that n is a perfect square, and n is a sum of two squares. Note that the integer $n(n + 1)$ can be expressed as a sum of two squares as

$$n(n + 1) = ((k^2 + 1)^2)^2 + (k^2 + 1)^2. \tag{2}$$

Using the identity

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2,$$

it follows that the integer $n(n + 1)$ can also be expressed as a sum of two squares as

$$n(n + 1) = ((k^2 - 1)(k^2 + 1) - 2k)^2 + (k^2 - 1 + 2k(k^2 + 1))^2,$$

that is, as

$$n(n + 1) = (k^4 - 2k - 1)^2 + (2k^3 + k^2 + 2k - 1)^2. \tag{3}$$

It suffices to find infinitely many positive integers k such that Eq. (2), Eq. (3) provide two different expressions of $n(n + 1)$ as a sum of two positive squares.

Since the polynomials $k^4 - 2k - 1, 2k^3 + k^2 + 2k - 1$ have degrees higher than the degree of the polynomial $k^2 + 1$, it follows that for large enough values of k, the inequalities

$$k^4 - 2k - 1 > k^2 + 1, 2k^3 + k^2 + 2k - 1 > k^2 + 1$$

hold. Indeed,

$$\begin{aligned} k^4 - 2k - 1 - (k^2 + 1) &= \left(\frac{k^4}{3} - k^2\right) + \left(\frac{k^4}{3} - 2k\right) + \left(\frac{k^4}{3} - 2\right) \\ &> 0, \\ 2k^3 + k^2 + 2k - 1 - (k^2 + 1) &= 2k^3 - 1 + 2k - 1 \\ &> 0 \end{aligned}$$

hold if $k \geq 2$. Hence, for any integer $k \geq 2$, putting $n = (k^2 + 1)^2$, it follows that $n(n + 1)$ can be expressed as the sum of two positive squares in at least two different ways. ■