Cubic polynomials

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Cubic polynomials

Example 1.1 (India RMO 2000 P2)**.** Solve the equation $y^3 = x^3 + 8x^2 - 6x + 8$, for positive integers x and y.

Solution 1. Let x, y be positive integers satisfying the given equation. Since x is a positive integer, it follows that $8x^2 - 6x + 8$ is positive, and hence, $y^3 \geq x^3$ holds. This shows that $y = x + k$ for some positive integer k. Substituting $y = x + k$ in the given equation and simplifying, we obtain

$$(3k - 8)x^2 + (3k^2 + 6)x + k^3 - 8 = 0.$$

Since x is positive, it follows that $k \leq 2$. If $k = 1$, then

$$5x^2 - 9x + 7 = 0$$

holds, which shows that $7 \equiv x(x + 1) \pmod 2$, which yields $7 \equiv 0 \pmod 2$, which is impossible. This implies that $k = 2$. It follows that

$$2x^2 - 18x = 0,$$

which gives $x = 9$, and consequently, we obtain $y = 9 + 2 = 11$.

Also note that if $(k, x) = (2, 9)$, then

$$(3k - 8)x^2 + (3k^2 + 6)x + k^3 - 8 = 0$$

holds, and it gives

$$(x + k)^3 = x^3 + 8x^2 - 6x + 8,$$

which shows that $(x, y) = (9, 11)$ is a solution to the given equation.

It follows that $(9, 11)$ is the only solution of the given equation in the positive integers. ■

> **Remark.** After arriving at the above solution, one can rewrite it to make it brief by observing that
>
> $$x^3 + 8x^2 - 6x + 8 - (x + 1)^3 = 5x^2 - 9x + 7,$$
>
> which is positive for any positive integer x (in fact, it is positive for any real number x), and then concluding that $y = x + k$ for some integer $k > 1$.

Example 1.2 (India RMO 2015f P5)**.** Solve the equation $y^3 + 3y^2 + 3y = x^3 + 5x^2 - 19x + 20$ for positive integers x, y.

Solution 2. Let x, y be positive integers satisfying the above equation. Note that the given equation can be rewritten as

$$(y+1)^3 = x^3 + 5x^2 - 19x + 21.$$

Note that

$$5x^2 - 19x + 21 > 5x^2 - 19x \geq 0$$

holds if $x \geq 4$. Also note that $5x^2 - 19x + 21 > 0$ if x lies in $\{1, 2, 3\}$. Since x is a positive integer, it follows that $5x^2 - 19x + 21 \geq 1$. This shows that $(y+1)^3 > x^3$, and hence, $y + 1 = x + k$ for some positive integer k. Note that

$$(y+1)^3 = x^3 + 5x^2 - 19x + 21$$

is equivalent to

$$3kx^2 + 3k^2x + k^3 = 5x^2 - 19x + 21,$$

which simplifies to

$$(3k-5)x^2 + (3k^2+19)x + (k^3 - 21) = 0.$$

Note that if $k \geq 2$, then using $x > 0$, we obtain

$$\begin{aligned}
&(3k-5)x^2 + (3k^2+19)x + (k^3 - 21) \\
&= (3k-5)x^2 + 3k^2x + 19(x-1) + k^3 - 2 \\
&> 1.
\end{aligned}$$

This shows that k is equal to 0 or 1. If $k = 0$, then $5x^2 - 19x + 21 = 0$ holds, which implies that 21 is even, which is impossible. This shows that $k = 1$, and hence $x = y$. It follows that

$$2x^2 - 22x + 20 = 0,$$

and hence x is equal to one of $1, 10$. Consequently, we obtain (x, y) is equal to $(1, 1)$ or $(10, 10)$.

Note that for any integers x, y, k satisfying $y + 1 = x + k$ and

$$3kx^2 + 3k^2x + k^3 = 5x^2 - 19x + 21,$$

, we have

$$(y+1)^3 = x^3 + 5x^2 - 19x + 21.$$

It follows that $(1, 1), (10, 10)$ also satisfy the given equation.

Consequently, the solutions of the given equation over the positive integers are precisely $(1, 1)$ and $(10, 10)$. ∎