Chinese remainder theorem

MOPSS

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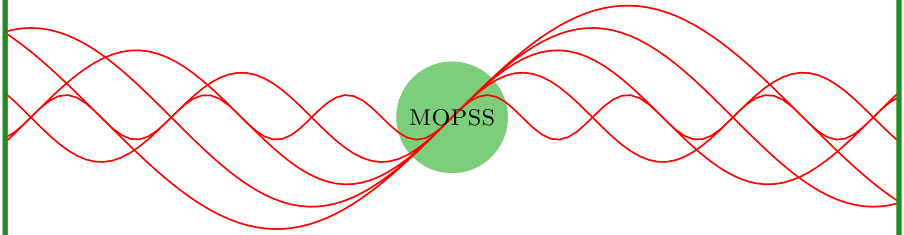
Mathematics Olympiad

Problem Solving Sessions



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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice **On reading solutions**, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's **Advice for writing proofs/Remarks on English**, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why **math olympiads are a valuable experience for high schoolers** in the post on **Lessons from math olympiads**, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.
- **Tips for writing up solutions** by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.

List of problems and examples

§1 Chinese remainder theorem

Example 1.1 (Canada CMO 1991 P1)**.** Show that the equation $x^2 + y^5 = z^3$ has infinitely many solutions in integers x, y, z for which $xyz \neq 0$.

> **Walkthrough** — Note that
>
> $$2^{10} + 2^{10} = 2^{11}$$
>
> holds. But this does not seem to be of any help! What about $2^{20} + 2^{20}$? Also note that if (x, y, z) is a solution, then another solution can be obtained by multiplying $x^2 + y^5 = z^3$ by k^{30}, namely, $(k^{15}x, k^6 y, k^{10} z)$ is also a solution.

Solution 1. Note that

$$(k^{15}2^{10})^2 + (k^6 2^4)^5 = k^{30}2^{21} = (k^{10}2^7)^3$$

holds for any positive integer k. ■

Example 1.2. Show that there are infinitely many triples (x, y, z) of positive integers such that $x^2 + y = z^7$.

> **Walkthrough** — Does $2^6 + 2^6 = 2^7$ help?

Solution 2. Note that $2^6 + 2^6 = 2^7$ holds. It shows that $(2^3k^7, 2^6k^{14}, 2k^2)$ is a solution to the given equation. This proves the result. ■

Here is another solution to the above problem.

Solution 3. Note that if x, y, z are positive integers satisfying $y = 2x + 1$ and $x^2 + y = z^7$, then $(x + 1)^2 = y^7$ holds, which shows that

$$x = a^7 - 1, z = a^2$$

for some positive integer a. Observe that for any integer $a \geq 2$, the triple $(a^7 - 1, 2(a^7 - 1) + 1, a^2)$ of positive integers satisfy the given equation. Thus the result follows. ■

Example 1.3. There are infinitely many triples (x, y, z) of positive integers such that $x^2 + y^3 = z^7$.

> **Walkthrough** — Note that $2^6 + 2^6 = 2^7$
>
> $$(2^3)^2 + (2^2)^3 = 2^7$$
>
> holds. This shows that $(2^3 k^{3\cdot 7}, 2^2 k^{2\cdot 7}, 2k^{2\cdot 3})$ is a solution to the above equation for any positive integer k.

Here is another argument from [Beu12].

Solution 4. Take three positive integers a, b, c satisfying $a + b = c$. Multiplying it by $a^{21}b^{14}c^6$, we obtain

$$(a^{11}b^7c^3)^2 + (a^7b^5c^2)^3 = (a^3b^2c)^7.$$

■

> **Remark.** The following is a result due to Poonen–Schaefer–Stoll [PSS07], determining the primitive solutions to $x^2 + y^3 = z^7$, that is, the triples of integers having prime factor in common and satisfying this equation. Its proof uses techniques from Arithmetic Geometry, and lies beyond the scope of this modest notes, for obvious reasons! The **only purpose** of stating the following result is to indicate that finding all solutions of prescribed nature to some equation (say, the primitive solutions to the above equation) often turns out to be a problem of considerable interest, and may require modern techniques to solve them.
>
> > **Theorem 1** (Poonen, Schaefer, Stoll)
> > The primitive solutions to $x^2 + y^3 = z^7$ are the 16 tuples
> >
> > $$(\pm 1, -1, 0), (\pm 1, 0, 1), \pm(0, 1, 1), (\pm 3, -2, 1), (\pm 71, -17, 2),$$
> >
> > $$(\pm 2213459, 1414, 65), (\pm 15312283, 9262, 113), (\pm 21063928, -76271, 17).$$

Example 1.4 (India RMO 2015a P3, Canada CMO 1991 P1 Example 1.1)**.** Show that there are infinitely many triples (x, y, z) of integers, such that $x^3 + y^4 = z^{31}$.

> **Walkthrough** —
>
> **(a)** Note that
>
> $$2^{12k} + 2^{12k} = 2^{12k+1}$$
>
> holds.
>
> **(b)** It suffices to find an positive integer k such that $12k + 1 = 31\ell$ holds

for some positive integer ℓ. This shows that any such integer ℓ satisfies $\ell \equiv 1 \bmod 3$, and $\ell \equiv -1 \bmod 4$, which implies that $\ell \equiv 7 \bmod 12$. Moreover, if ℓ is congruent to 7 modulo 12, then

$$31\ell - 1 \equiv 31 * 7 - 1 \bmod 12 \equiv 7^2 - 1 \bmod 12 \equiv 0 \bmod 12$$

holds, or equivalently, there exists an integer k_ℓ such that

$$12k_\ell + 1 = 31\ell$$

holds.

(c) Note that

$$(2^{4k_\ell}, 2^{3k_\ell}, 2^\ell)$$

satisfies the given equation, for any positive integer $\ell \equiv 7 \bmod 12$.

Remark. Alternatively, one may argue as follows, motivated by the solution of Example 1.3.

Solution 5. Let a, b, c be integers such that $a + b = c$. Multiplying by $a^p b^q c^r$, we get

$$a^{p+1}b^q c^r + a^p b^{q+1} c^r = a^p b^q c^{r+1}.$$

Note that $a^{p+1}b^q c^r, a^p b^{q+1} c^r, a^p b^q c^{r+1}$ are a 3rd, 4th, and a 31st power respectively if 3 divides $p+1, q, r$, 4 divides $p, q+1, r$, 31 divides $p, q, r+1$, which holds if there are positive integers P, Q, R such that

$$p = 4 \cdot 31 \cdot P, q = 3 \cdot 31 \cdot Q, r = 3 \cdot 4 \cdot R$$

holds, and 3 divides $4 \cdot 31 \cdot P + 1$, 4 divides $3 \cdot 31 \cdot Q + 1$, 31 divides $3 \cdot 4 \cdot R + 1$. Since the integers $3, 4, 31$ are pairwise coprime, it follows that each of them is coprime to the product of the remaining two integers, and hence, by the division algorithm, there are positive integers P, Q, R such that 3 divides $4 \cdot 31 \cdot P + 1$, 4 divides $3 \cdot 31 \cdot Q + 1$, 31 divides $3 \cdot 4 \cdot R + 1$, or equivalently,

$$P \equiv 2 \bmod 3, Q \equiv 3 \bmod 4, R \equiv 18 \bmod 31$$

holds. In the above, the last congruence is obtained using $(3 \cdot 4) \cdot (6 \cdot 3) \equiv -1 \bmod 31$, which follows from $3 \cdot 4 \cdot 6 \equiv 10 \bmod 31$ and $10 \cdot 3 \equiv -1 \bmod 31$.

This shows that for any positive integers P, Q, R satisfying the above congruence conditions (for instance, $P = 2, Q = 3, R = 18$), and for any positive integers a, b, c satisfying $a + b = c$, the triple

$$((a^{p+1}b^q c^r)^{\frac{1}{3}}, (a^p b^{q+1} c^r)^{\frac{1}{4}}, (a^p b^q c^{r+1})^{\frac{1}{31}})$$

satisfies the given equation, where $p = 124P, q = 93Q, r = 12R$. Hence the given equation has infinitely many solution in integers. ■

> **Remark.** All of the above examples (which excludes Theorem 1!) admit solutions using the Chinese remainder theorem.

References

[**Beu12**] Frits Beukers. "The generalized Fermat equation". In: *Explicit methods in number theory.* Vol. 36. Panor. Synthèses. Soc. Math. France, Paris, 2012, pp. 119–149 (cited p. 3)

[**PSS07**] Bjorn Poonen, Edward F. Schaefer, and Michael Stoll. Twists of $X(7)$ and primitive solutions to $x^2+y^3 = z^7$. In: *Duke Math. J.*, **137**:1 (2007), pp. 103–158. issn: 0012-7094. doi: 10.1215/S0012-7094-07-13714-1. url: http://dx.doi.org/10.1215/S0012-7094-07-13714-1 (cited p. 3)