Binomial coefficients

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Binomial coefficients

Example 1.1 (AIME 1983 P8, India Pre-RMO 2015 P10(?))**.** Determine the largest 2-digit prime factor of the integer $\binom{200}{100}$.

Solution 1. The highest power of any two digit prime p dividing $(100!)^2$ is at least p^2, and hence, if p is a two digit prime that divides $\binom{200}{100}$, then p^3 divides $200!$, or equivalently, $3p \leq 200$, which implies that $p \leq 66$, and this gives $p \leq 61$.

Note that 61 is a primes, its highest power dividing $(100!)^2$ is 61^2. Using $3 \cdot 61 < 200$, it follows that 61^3 divides $200!$. Hence, 61 divides $\binom{200}{100}$.

This shows that 61 is the largest 3-digit prime factor of $\binom{200}{100}$. ■

Example 1.2 (India RMO 1992 P3)**.** Determine the largest 3-digit prime factor of the integer $\binom{2000}{1000}$.

Solution 2. Suppose there is a 3-digit prime p which divides the integer $\binom{2000}{1000}$. Note that

$$\binom{2000}{1000} = \frac{2000 \cdot 1999 \cdot 1998 \cdot \dots \cdot 1001}{1000 \cdot 999 \cdot 998 \cdot \dots \cdot 1}$$

holds. If $p > 500$, then using the fact that p divides $\binom{2000}{1000}$, it follows that $3p \leq 2000$, which implies $p \leq 666$, and hence, $p \leq 661$. This shows that any 3-digit prime divisor of $\binom{2000}{1000}$, larger than 500, is at most 661.

Note that 661 is a 3-digit prime, and it divides $1000!$. Since $1000 > 661 > 500$ holds, it follows that $2000 > 2 \cdot 661 \geq 1001$. Also note that $3 \cdot 661 < 2000$. This implies that the highest power of 661 dividing $1000!$ is 661, and the highest power of 661 dividing $2000 \cdot 1999 \cdot 1998 \cdot \dots \cdot 1001$ is at least 661^2. Consequently, 661 divides $\binom{2000}{1000}$.

This proves that 661 is the largest 3-digit prime factor of the integer $\binom{2000}{1000}$. ■