Problem Set

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Contents

Instructions

- There are a few examples discussed, and a few problems have been provided.
- Solutions to the following problems are to be submitted.
  - Problem 1.1
  - Problem 1.2
  - Problem 2.1
  - Problem 2.2
  - Problem 3.1
- You are encouraged to go through the examples discussed along with
  - Evan Chen's advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.

  This may prove to be useful while solving the problems.
- Please make sure that the solutions are neatly written.
  - You are encouraged to go through[1] Evan Chen's *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- If you do not have a complete solution to a problem, then you may indicate the progress that you have made, and include the details.
- Solving all the problems is **NOT** mandatory. However, we would like to know what you have thought about the problems.

[1] One may also take a look at the post on *Lessons from math olympiads* by Evan Chen, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/, where he discusses why *math olympiads are a valuable experience for high schoolers*.

- The deadline and the other relevant details have been mentioned at https://jpsaha.github.io/MOTP/MOPSS/.
- For any further questions regarding the Mathematics Olympiad Problem Solving Sessions (MOPSS), please get in touch with
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§1 Sum of squares

§1.1 Adding two perfect squares

The squares of the nonnegative integers

$$0, 1, 2, 3, 4, 5, \ldots$$

are called the **perfect squares**. So, the perfect squares are

$$0, 1, 4, 9, 16, 25, 36, 49, \ldots.$$

Example 1.1. Show that the positive integers of the form $4n + 3$, that is, the integers

$$3, 7, 11, 15, 19, \ldots$$

cannot be written as the sum of two perfect squares.

Summary — Show that the squares leave a remainder of 0 or 1 upon division by 4. Conclude that a sum of two squares leaves a remainder of $0, 1, 2$ upon division by 4.

Walkthrough —

(a) Consider the integers

$$\begin{gathered}0^2 + 1^2, 0^2 + 2^2, 0^2 + 3^2, 0^2 + 4^2, \ldots, \\ 1^2 + 1^2, 1^2 + 2^2, 1^2 + 3^2, 1^2 + 4^2, \ldots, \\ 2^2 + 1^2, 2^2 + 2^2, 2^2 + 3^2, 2^2 + 4^2, \ldots, \\ 3^2 + 1^2, 3^2 + 2^2, 3^2 + 3^2, 3^2 + 4^2, \ldots, \\ 4^2 + 1^2, 4^2 + 2^2, 4^2 + 3^2, 4^2 + 4^2, \ldots.\end{gathered}$$

(b) Observe that upon division by 4, they leave the integers $0, 1, 2$ as remainders.

$$0^2 + 1^2 \rightsquigarrow \mathbf{1}, 0^2 + 2^2 \rightsquigarrow \mathbf{0}, 0^2 + 3^2 \rightsquigarrow \mathbf{1}, 0^2 + 4^2 \rightsquigarrow \mathbf{0}, \ldots,$$

$$1^2+1^2 \rightsquigarrow \mathbf{2}, 1^2+2^2 \rightsquigarrow \mathbf{1}, 1^2+3^2 \rightsquigarrow \mathbf{2}, 1^2+4^2 \rightsquigarrow \mathbf{1}, \ldots,$$
$$2^2+1^2 \rightsquigarrow \mathbf{1}, 2^2+2^2 \rightsquigarrow \mathbf{0}, 2^2+3^2 \rightsquigarrow \mathbf{1}, 2^2+4^2 \rightsquigarrow \mathbf{0}, \ldots,$$
$$3^2+1^2 \rightsquigarrow \mathbf{2}, 3^2+2^2 \rightsquigarrow \mathbf{1}, 3^2+3^2 \rightsquigarrow \mathbf{2}, 3^2+4^2 \rightsquigarrow \mathbf{1}, \ldots,$$
$$4^2+1^2 \rightsquigarrow \mathbf{1}, 4^2+2^2 \rightsquigarrow \mathbf{0}, 4^2+3^2 \rightsquigarrow \mathbf{1}, 4^2+4^2 \rightsquigarrow \mathbf{0}, \ldots.$$

(c) Show that it is **always** the case, namely, upon division by 4, the sum of two perfect squares leaves one of $0, 1, 2$ as the remainder.

(d) Conclude that no integer, which leaves the remainder of 3 **upon division by** 4, can be written as the sum of two squares.

Solution. The solution relies on the following Claim.

Claim — For any integer x, the integer x^2 leaves a remainder of 0 or 1 upon division by 4.

Proof of the Claim. Let x be an integer. Let us consider the following cases.

1. Upon division by 4, the integer x leaves a remainder of 0.
2. Upon division by 4, the integer x leaves a remainder of 1.
3. Upon division by 4, the integer x leaves a remainder of 2.
4. Upon division by 4, the integer x leaves a remainder of 3.

In the first case, x is a multiple of 4, and hence x^2 leaves a remainder of 0 upon division by 4. Similarly, in the third case, x is a multiple[2] of 2, that is, x is equal to $2k$, and hence x^2 is a multiple of 4.

In the second case, x is equal to $4k+1$ for some integer k. Note that

$$\begin{aligned} x^2 &= (4k+1)^2 \\ &= (4k)^2 + 2 \cdot 4k + 1 \\ &= 4(4k^2+2k)+1, \end{aligned}$$

and hence x^2 leaves a remainder of 1 upon division by 4.

In the fourth case, x is equal to $4k+3$ for some integer k. Note that

$$\begin{aligned} x^2 &= (4k+3)^2 \\ &= (4k)^2 + 2 \cdot 4k \cdot 3 + 9 \\ &= 4(4k^2+6k+2)+1, \end{aligned}$$

and hence x^2 leaves a remainder of 1 upon division by 4.

This proves the Claim. □

[2] Is it clear?

Using the Claim, it follows that a sum of two squares leaves one of $0, 1, 2$ as a remainder upon division by 4. Hence, no integer of the form $4n + 3$ can be expressed as a sum of two perfect squares. ■

Problem Statement 1.1

Let m, n be two positive integers. If each of them can be expressed as a sum of two perfect squares, then show that their product mn can also be expressed as a sum of two perfect squares.

In other words, if there are nonnegative integers a, b, c, d such that

$$m = a^2 + b^2, \quad n = c^2 + d^2,$$

then prove that there are integers x, y such that

$$mn = x^2 + y^2.$$

Hint — Consider the product $(a^2 + b^2)(c^2 + d^2)$, and try to find out suitable integers x, y such that

$$(a^2 + b^2)(c^2 + d^2) = x^2 + y^2.$$

§1.2 Adding three perfect squares

Problem Statement 1.2

Show that the positive integers of the form $8n + 7$, that is, the integers

$$7, 15, 23, 31, 39, 47, \ldots$$

cannot be written as the sum of three perfect squares.

Hint —

- Show that any square leaves one of $0, 1, 4$ as a remainder upon division by 8.
- Conclude that a sum of three squares leaves one of $0, 1, 2, 3, 4, 5, 6$ as a remainder upon division by 8.

- Does proceeding along the lines of Example 1.1 help?

§2 Rational numbers

§2.1 Summing the reciprocals of the positive integers

Let us consider the sum of the reciprocals of the first few positive integers. If we consider

$$1 + \frac{1}{2},$$

we obtain $\frac{3}{2}$. Next, if we include two more terms, we get

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4},$$

which is equal to $\frac{25}{12}$. Adding more terms would increase the sum. An immediate question that may occur is the following.

Question
- Can we determine by what amount this sum grows?
- Can this sum be larger than a given number, for instance, 1000, after we have added enough terms to it?

Note that the terms that we are introducing are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$, which become **increasingly closer to zero**.

Question

Can enough gradually decreasing numbers (say, decreasing to zero) be added together so that their sum is larger than any given number, for example, 10^{10}?

Let us go back to the our prior activity of adding more and more reciprocals of positive integers. If we consider four more terms, and consider

$$\begin{aligned} 1 &+ \frac{1}{2} \\ &+ \frac{1}{3} + \frac{1}{4} \\ &+ \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}, \end{aligned}$$

then one may observe that there is a **growth**. To be specific, one may note that

$$1 + \frac{1}{2}$$

$$
\begin{aligned}
&+\frac{1}{3}+\frac{1}{4} \\
&+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} \\
>1&+\frac{1}{2} \\
&+\frac{1}{4}+\frac{1}{4} \\
&+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} \\
>1&+\frac{1}{2} \\
&+\frac{2}{4} \\
&+\frac{4}{8} \\
>1&+\frac{3}{2}.
\end{aligned}
$$

Continuing the same argument, one may observe that

$$
\begin{aligned}
1&+\frac{1}{2} \\
&+\frac{1}{3}+\frac{1}{4} \\
&+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} \\
&+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16} \\
>1&+\frac{4}{2}.
\end{aligned}
$$

In fact, one can prove the following.

Lemma

The sum of the reciprocals of the integers between 1 and 2^n is at least as large as $1+\frac{n}{2}$, that is,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2^n}\geq 1+\frac{n}{2}.$$

Hence, given any number, say 10^{100}, one can add the reciprocals of enough positive integers (for instance, those between 1 and $2^{2\cdot 10^{100}}$) to obtain a sum larger than 10^{100}.

Question

What about getting an integer as such a sum?

Let us consider the problem below.

Problem Statement 2.1

Let n be a positive integer. Show that the sum of the reciprocals of the integers between 1 and 2^n, that is, the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n - 1} + \frac{1}{2^n}$$

is not an integer.

Hint —

(a) Try to see what would happen if such a sum were equal to an integer k, that is,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n} = k$$

holds some for integer k.

(b) Then we would have

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n - 1} = k - \frac{1}{2^n}.$$

(c) Does an argument using the method of adding fractions (after taking lcm of the denominators etc.) help?

(d) Does comparing the lcms of the denominators of the fractions of both the sides help? Do you obtain a contradiction?

(e) As usual, first working with a few specific values of n (for instance, $n = 3, 4, 5, 6$ etc.) may help to gain an insight to work out the general case (that is, for any n, without resorting to taking specific values of n).

§2.2 Summing the reciprocals of the squares of the positive integers

Question

What would happen if we add the reciprocals of the nonzero perfect squares? Can enough such perfect squares be added so that the sum becomes larger than a given quantity?

The following problem addresses this question.

Problem Statement 2.2

Show that for any positive integer n, the sum of the reciprocals of the squares of the integers between 1 and n is smaller that 2, that is,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2.$$

Remark. First, notice the apparent similarity of the above problem with what we discussed at the beginning of Section 2.1, where we obtained a **lower bound** for the sum of the reciprocals of the first few positive integers, by cleverly replacing the denominators by suitably chosen **larger** integers. For the problem above, one may follow the same strategy. More specifically, to obtain an **upper bound** for the sum of the reciprocals of the squares of the first few positive integers, one may replace the denominators by suitably chosen **smaller** integers. Next, in the former case, after replacing the denominators by suitably chosen **larger** integers, certain convenient **additions** led to a lower bound. For the above problem, one may hope that after replacing the denominators by suitably chosen **smaller** integers, certain convenient **cancellations** would yield an upper bound.

Hint —

(a) As usual, first working with a few specific values of n (for instance, $n = 3, 4, 5, 6$ etc.) may help to gain an insight to work out the general case (that is, for any n, without resorting to taking specific values of n).

(b) Do the inequalities

$$\frac{1}{2^2} < \frac{\mathbf{1}}{1 \cdot 2},$$
$$\frac{1}{3^2} < \frac{\mathbf{1}}{2 \cdot 3},$$
$$\frac{1}{4^2} < \frac{\mathbf{1}}{3 \cdot 4},$$

$$\frac{1}{5^2} < \frac{\mathbf{1}}{4 \cdot 5},$$

etc. help?

(c) Can you express the blue **1**'s in the numerator of each of these rationals as a difference of suitable consecutive integers?

§3 Tiling

Example 3.1. A domino is a 2×1 rectangle. For what integers m and n, can one cover an $m \times n$ rectangle with non-overlapping dominoes?

Walkthrough —

(a) If an $m \times n$ rectangle admits a covering by non-overlapping dominos, then show that at least one of the integers m, n has to be even.

(b) If at least one of m, n is even, then prove that an $m \times n$ rectangle admits a covering by non-overlapping dominos.

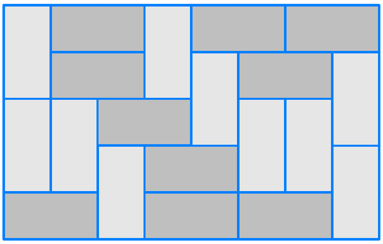

Figure 1: A tiling of a 5×8 rectangle with non-overlapping dominoes, Example 3.1

Solution. In the following, an $m \times n$ rectangle is to be thought as an $m \times n$ rectangular grid.

To be able to cover an $m \times n$ rectangle by non-overlapping dominoes, it is necessary for the product mn to be even, and hence, at least one of m, n is even. Indeed, if an $m \times n$ rectangle admits a covering using k non-overlapping dominoes (for instance, as in Fig. 1 with $m = 5$, $n = 8$ and $k = 20$), then those dominoes together cover $2k$ unit squares, and this yields that $2k = mn$. Hence, at least one of m, n is even.

Moreover, when at least one of m, n is even, an $m \times n$ rectangle can be covered by non-overlapping dominoes by covering each row by $m/2$ (resp. each column by $n/2$) non-overlapping dominos if m (resp. n) is even.

This shows that an $m \times n$ rectangle can be covered by non-overlapping dominoes if and only if at least one of m, n is even. ■

Remark. The above conclusion shows that an $m \times n$ rectangle admits a covering by non-overlapping dominoes if and only if it admits a covering by non-overlapping dominoes in the **most obvious manner**, that is, a covering by non-overlapping dominoes such that all of them are either horizontal or vertical.

Problem Statement 3.1

Show that an $m \times n$ rectangle admits a covering by non-overlapping 3×1 rectangles if and only if 3 divides m or 3 divides n.

Hint — Does thinking along the lines of Example 3.1 help?