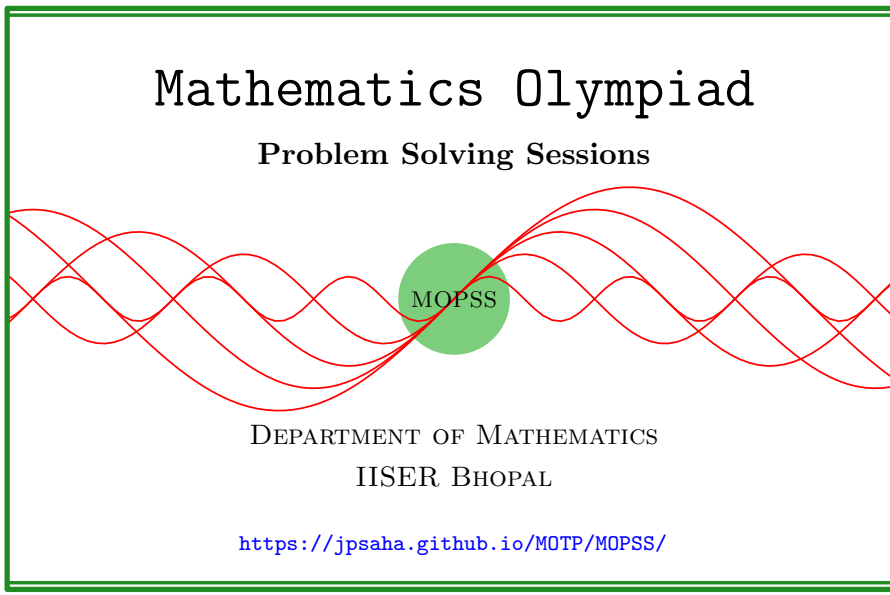
MOPSS

24 October 2026



Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

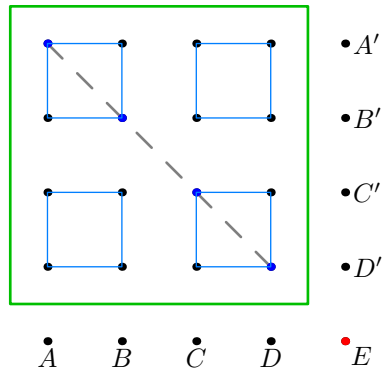


Figure 1: RMO 2018a P4, Exercise 1

Exercise 1 (RMO 2018a P4,). Let E denote the set of 25 points (m, n) in the xy-plane, where m, n are natural numbers, $1 \leq m \leq 5, 1 \leq n \leq 5$. Suppose the points of E are arbitrarily coloured using two colours, red and blue. Show that there always exist four points in the set E of the form $(a, b), (a + k, b), (a + k, b + k), (a, b + k)$ for some positive integer k such that at least three of these four points have the same colour. (That is, there always exist four points in the set E which form the vertices of a square and having at least three points of the same colour.)

Walkthrough —

(a) Assume that the conclusion is false, and by interchanging the colors (if necessary), assume that there are more red points than the blue ones.

(b) Show that one of the four corners is red, and assume without loss of generality that the point E (as in Fig. 1) is red.

(c) Considering the number of red points in each of the sets $\{A, A'\}$, $\{B, B'\}$, $\{C, C'\}$, $\{D, D'\}$, prove that these four sets together contain at most 4 red points, and conclude that the green square (as in Fig. 1) contains at least 8 red points.

(d) Consider the blue squares in the green square, to show that the green square contains exactly 8 red points.

(e) Conclude that there are exactly 13 red points, and each of the sets $\{A, A'\}$, $\{B, B'\}$, $\{C, C'\}$, $\{D, D'\}$ contains exactly one red point, and the green square contains exactly 8 red points.

(f) For each point on the dashed diagonal (as in Fig. 1), consider the square having this point and E as the endpoints of one of its diagonals to show that the points on the dashed diagonal are blue.

(g) Note that each point within the green square lying outside the dashed diagonal, is a vertex of a square having two of its remaining vertices on

the dashed diagonal. Conclude that each such point is red.

(h) Obtain a contradiction to conclude that the assumption is false.

Solution 1. On the contrary, let us assume that there is no axes-parallel square having at least three vertices of the same color.

Note that at least 13 among those 25 points are of the same color. Without loss of generality, assume that those are red. By our hypothesis, it follows that among the four vertices at the corners, there is at least one red vertex. Without loss of generality[1], let us assume that the bottom-right vertex is red, and denote this vertex by E (as in Fig. 1).

Note that each of the sets $\{A, A'\}, \{B, B'\}, \{C, C'\}, \{D, D'\}$ (as in Fig. 1) contains at most one red point, otherwise, we can form an axes-parallel square with at least three vertices of the same color. Consequently, the green square (as in Fig. 1) contains at least $13 - 5 = 8$ red points.

If the green square contains at least 9 red points, then at least one of the four blue squares (as in Fig. 1) contains at least three red points, which is not the case by our hypothesis. Hence, the green square contains exactly 8 red points. Consequently, there are precisely 13 red points among the 25 points, and each of the sets $\{A, A'\}, \{B, B'\}, \{C, C'\}, \{D, D'\}$ contains exactly one red point.

For any given point on the dashed diagonal (as in Fig. 1), consider the square having this point and E as the endpoints of one of its diagonals. Note that one of the other two vertices of this square is red. By our hypothesis, it follows that all the points on the gray diagonal (as in Fig. 1) are blue.

Note that each point within the green square lying outside the dashed diagonal, is a vertex of a square having two of its remaining vertices on the dashed diagonal. This shows that all the points within the green square, lying outside the gray diagonal, are red. Hence, there exists an axes-parallel square, all whose vertices are red. This contradicts our hypothesis, and hence, it follows that there exists an axes-parallel square having at least three vertices of the same color. ■

Exercise 2 (RMO 2023b P6,). Consider a set of 16 points arranged in a 4×4 square grid formation. Prove that if any 7 of these points are coloured blue, then there exists an isosceles right-angled triangle whose vertices are all blue.

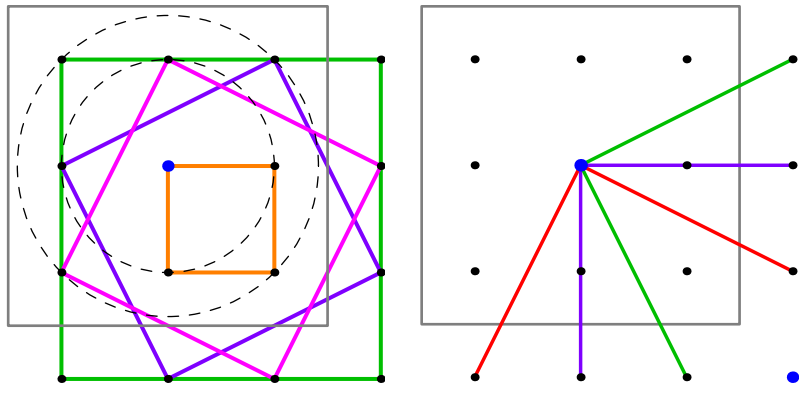
The following is from AoPS, is due to Rohan (Goyal?) as mentioned here by L567.

Walkthrough —

(a) Show that if the small square (as in Fig. 2a) does not contain a blue point, then we are done.

(b) It remains to consider the case when the small square (as in Fig. 2a)

[1] Convince yourself that there is **indeed no loss of generality** in assuming so.



(a) (b)

Figure 2: RMO 2023b P6, Exercise 2

contains at least one blue point.

(c) Rotating the configuration about the center of the small square (if necessary), assume that the top-left vertex of the small square (as in Fig. 2a) is blue.

(d) Prove that the gray square contains at most three blue points.

- Consider the case when at least two blue points lie on the bigger dashed circle. Show that the smaller dashed circle does not contain any blue point, in this case. Hence, the gray square contains at most three blue points.
- Similarly, if the smaller dashed circle contains at least two blue points, then the gray square (as in Fig. 2b) contains at most three blue points.

(e) It suffices to consider that each one of the red, purple, and green L-shapes, has at most one end-point which is blue (otherwise, we are done).

(f) Use the above to show that the bottom-right point (as in Fig. 2b) is blue.

(g) Consider the point at the bottom-right corner, and the center of the gray square, and a blue end-point of an L-shape, to show that these three points form the vertices of an isosceles triangle having the required properties.

Solution 2. Note that the 16 points are the vertices of the four squares as in Fig. 2a. If no vertex of the small square is blue, then by the pigeonhole principle, at least one of the remaining three squares has at least three blue vertices, and hence there exists an isosceles right-angled triangle with blue vertices.

Let us assume that the small square (as in Fig. 2a) has a blue vertex. Rotating the configuration about the center of the small square (if necessary), we may

and do assume that the top-left vertex of the small square is blue. Henceforth, on the contrary, let us assume that there are no isosceles right-angled triangles with blue vertices.

> **Claim** — The gray square (as in Fig. 2a) contains at most three blue points.

Proof of the Claim. Note that the points within the gray square lies on the two dashed circles (as in Fig. 2a). Therefore, to prove the Claim, it suffices to show that if one of the dashed circles (as in Fig. 2a) contains at least two blue points, then the other dashed circle does not contain any blue point.

If at least two blue points lie on the bigger dashed circle, then using the assumption, it follows that no more blue points lies on it, and hence these two blue points lie along a diameter. It follows that no blue point lies on the other dashed circle.

If at least two blue points lie on the smaller dashed circle, then using a similar argument, it follows that no blue point lies on the bigger dashed circle.

The Claim follows. □

Note that if both the end-points of one of the red, purple, and green L-shapes (as in Fig. 2b) are blue, then these points together with the center of gray square form the vertices of an isosceles right-angled triangle, contradicting the assumption. Hence, each one of these three L-shapes has at most one end-point which is blue. Since there are 7 blue points, using the Claim, it follows that the bottom-right point is blue. Note that the center of the gray square, the bottom-right point, and a blue end-point of the "purple" L-shape, form the vertices of a triangle having the required properties. ■

Exercise 3 (British Mathematical Olympiad Round 1 2015/16 P4)**.** James has a red jar, a blue jar and a pile of 100 pebbles. Initially both jars are empty. A move consists of moving a pebble from the pile into one of the jars or returning a pebble from one of the jars to the pile. The numbers of pebbles in the red and blue jars determine the state of the game. The following conditions must be satisfied:

(a) The red jar may never contain fewer pebbles than the blue jar;

(b) The game may never be returned to a previous state.

What is the maximum number of moves that James can make?

> **Walkthrough** — Suppose there are n pebbles at the beginning, where n is even.
>
> **(a)** Note that the state can be thought as a pair (r, b) where r and b denote the number of pebbles in the red and the blue jar respectively, at some instance.

(b) Observe that $r \geq b$ and $r + b \leq n$.

(c) This shows that (r, b) lies in a specific triangular region on the plane.

(d) Conclude that the maximum number of moves that James can make is at most the total number of lattice points in this triangular region, which is equal to

$$T := \sum_{r=0}^{n} \sum_{b=0}^{\min\{r, n-r\}} 1 = 1 + 3 + 5 + \cdots + (n+1) = (n/2+1)^2.$$

(e) Show that there is a path in this triangular region beginning at the origin, and traversing through at least

$$T - \frac{n}{2}$$

points in this region. Does Figs. 3 and 4 help?

(f) Observe that each move changes the parity of the sum of the coordinates.

(g) Show that after make any number of moves, the number of states with an even sum of coordinates is at most one more than the number of states with an odd sum of coordinates.

(h) Prove that the number of points in the triangular region with an even sum of coordinates is precisely $n/2+1$ more than those with an odd sum of coordinates. See Figs. 5 and 6.

(i) If James makes an odd number of moves, say $2k-1$ moves, then show that the number of points on the path formed is

$$2k \leq 2O = T - n/2 - 1.$$

(j) Suppose James makes an even number of moves, say, $2k$ moves. Conclude that the number of points on the path formed is

$$k + k + 1 \leq 2O + 1 = T - n/2 - 1 + 1 = T - n/2.$$

(k) Show that the maximum number of moves that James can make is

$$T - n/2 - 1 = (n/2+1)^2 - n/2 - 1 = \frac{n^2 + 2n}{4}.$$

Exercise 4 (INMO 2015 P6,). From a set of 11 square integers, show that one can choose 6 numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that

$$a^2 + b^2 + c^2 \equiv e^2 + f^2 + g^2 \pmod{12}.$$

Walkthrough —

(a)

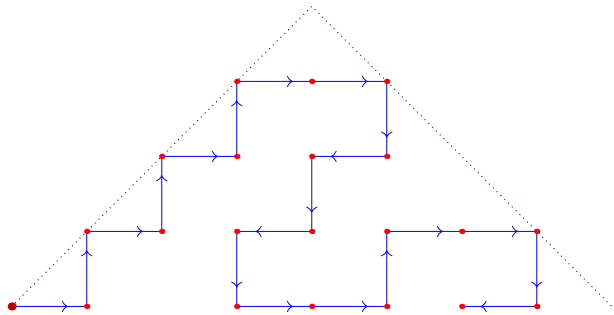


Figure 3: British Mathematical Olympiad Round 1 2015/16 P4, Exercise 3

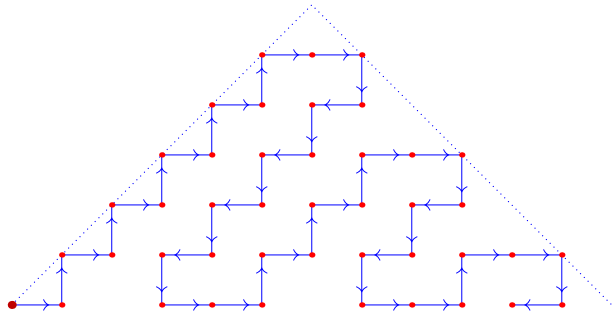


Figure 4: British Mathematical Olympiad Round 1 2015/16 P4, Exercise 3

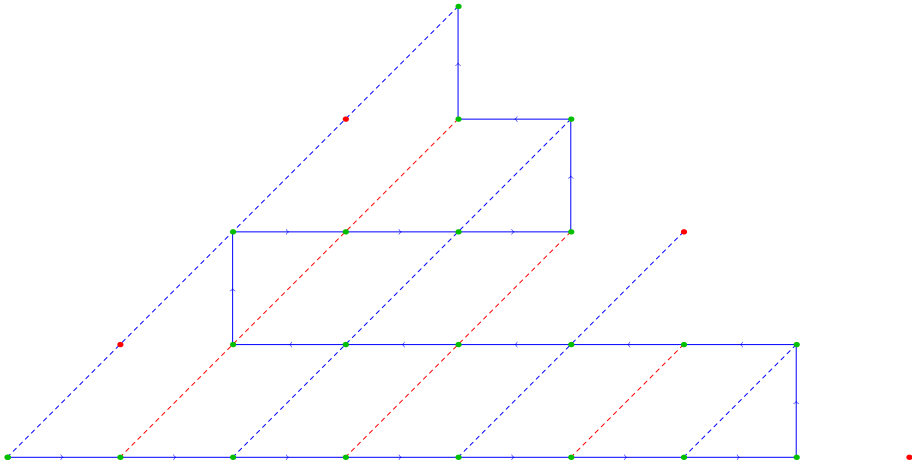


Figure 5: British Mathematical Olympiad Round 1 2015/16 P4, Exercise 3

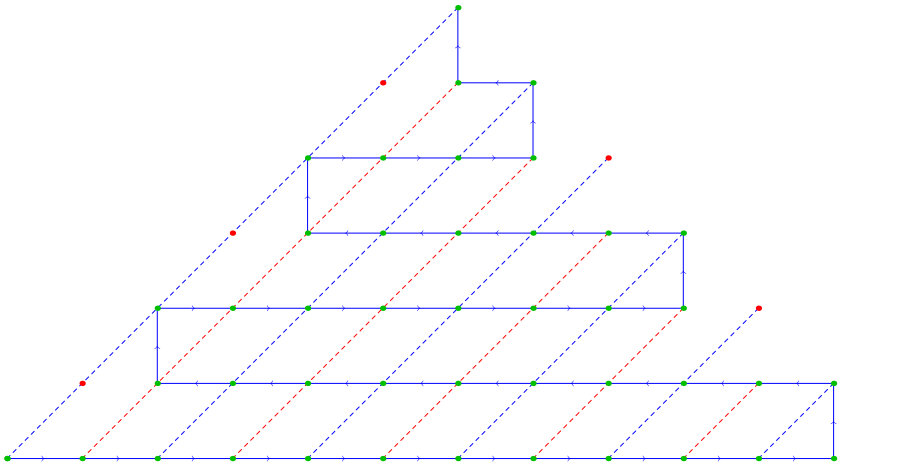


Figure 6: British Mathematical Olympiad Round 1 2015/16 P4, Exercise 3

Solution 3. Let A denote the set of the given 11 square integers. Let n denote the largest positive integer such that some n elements of A are congruent to each other modulo 12. Let X_n denote the set formed by some such n integers. Let x be an element of $\{0, 1, 4, 9\}$ which is congruent to any element of X_n modulo 12. If $n = 6$, then sum of any three of those 6 integers is congruent to the remaining three integers modulo 12. Note that any perfect square is congruent to one of $0, 1, 4, 9$ modulo 12.

Now let us consider the case that $n = 4$ or $n = 5$. Note that any element of $A \setminus X_n$ is congruent to some element of $\{0, 1, 4, 9\} \setminus \{x\}$ modulo 12. Since $A \setminus X_n$ has $11 - n \geq 6$ elements, it follows by the pigeonhole principle that some two elements x, y of $A \setminus X_n$ are congruent modulo 12. Note that the sum of x and any two elements of X_n is congruent to the sum of y and any two of the remaining elements of X_n.

It remains to consider the case that $n \leq 3$, which we assume from now on. Note that $n \geq 3$, since the set A has 11 elements and there are only four quadratic residues modulo 12. This shows that $n = 3$. Hence, some two elements of A are congruent to an element s of $\{0, 1, 4, 9\}$ modulo 12, and for any element t of $\{0, 1, 4, 9\} \setminus \{s\}$, some three elements a_t, b_t, c_t of A are congruent to t modulo 12. It follows that

$$\sum_{t \in \{0,1,4,9\} \setminus \{s\}} a_t \equiv \sum_{t \in \{0,1,4,9\} \setminus \{s\}} b_t \pmod{12}.$$

This completes the proof. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)