MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [Che25, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

Exercise 1 (IMOSL 2024 A1, AoPS, proposed by Santiago Rodríguez, Colombia)**.** Determine all real numbers α such that the number

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$

is a multiple of n for every positive integer n. (Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

Solution 1. Let α be a real number such that the number

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$

is a multiple of n for every positive integer n. Write $\alpha = k + \varepsilon$ with $k \in \mathbb{Z}$ and $0 \leq \varepsilon < 1$.

Claim — If $\varepsilon > 0$, then the inequality $\frac{1}{2} \leq \varepsilon$ holds.

Proof of the claim. Let us assume that $\varepsilon > 0$. Let r denote the smallest positive integer such that

$$\frac{1}{r} \leq \varepsilon.$$

Note that $r \geq 2$, and

$$\frac{1}{r} \leq \varepsilon < \frac{1}{r-1}$$

holds. Note that

$$\begin{aligned} &\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor \\ &= (k + 2k + \cdots + nk) + (\lfloor \varepsilon \rfloor + \lfloor 2\varepsilon \rfloor + \cdots + \lfloor n\varepsilon \rfloor) \\ &= \frac{kn(n+1)}{2} + (\lfloor \varepsilon \rfloor + \lfloor 2\varepsilon \rfloor + \cdots + \lfloor n\varepsilon \rfloor) \end{aligned}$$

holds for any positive integer n. Note that

$$\lfloor \varepsilon \rfloor = \lfloor 2\varepsilon \rfloor = \cdots = \lfloor (r-1)\varepsilon \rfloor = 0,$$

and

$$\lfloor r\varepsilon \rfloor = 1$$

holds. It follows that r divides the integer

$$\frac{kr(r+1)}{2} + (\lfloor \varepsilon \rfloor + \lfloor 2\varepsilon \rfloor + \cdots + \lfloor r\varepsilon \rfloor) = \frac{kr(r+1)}{2} + 1.$$

If r is odd, then it divides 1, which is impossible since $r > 1$. If r is even, then $2r$ divides $kr(r+1) + 2$, and in particular, r divides 2, implying $r = 2$. This proves the claim. $\square$

Claim — If $\varepsilon > 0$, then we have $\varepsilon \geq \frac{2}{3}$.

Proof. Assume that $\varepsilon > 0$. By the previous claim, we have $\varepsilon \geq \frac{1}{2}$. By hypothesis, 3 divides the integer

$$\frac{3k(3+1)}{2} + (\lfloor \varepsilon \rfloor + \lfloor 2\varepsilon \rfloor + \lfloor 3\varepsilon \rfloor) = 6k + 0 + 1 + \lfloor 3\varepsilon \rfloor.$$

Since $\frac{1}{2} \leq \varepsilon < 1$, we have $\frac{3}{2} \leq 3\varepsilon < 3$, and thus $\lfloor 3\varepsilon \rfloor$ is equal to 1 or 2. Using 3 divides $1 + \lfloor 3\varepsilon \rfloor$, we obtain $\lfloor 3\varepsilon \rfloor = 2$, and thus $\varepsilon \geq \frac{2}{3}$. □

Claim — Let r be an even positive integer such that $\varepsilon \geq \frac{r}{r+1}$. Then the inequality

$$\varepsilon \geq \frac{r+2}{r+3}$$

holds.

Proof. Note that for every integer $0 \leq \ell \leq r$, we have

$$\frac{\ell}{\ell+1} \leq \frac{r}{r+1} < 1,$$

and thus

$$\ell \leq \lfloor (\ell+1)\varepsilon \rfloor < \ell + 1,$$

which implies $\lfloor (\ell+1)\varepsilon \rfloor = \ell$. It follows that

$$\begin{aligned}
&\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor (r+1)\alpha \rfloor \\
&= (k + 2k + \cdots + (r+1)k) + (\lfloor \varepsilon \rfloor + \lfloor 2\varepsilon \rfloor + \cdots + \lfloor (r+1)\varepsilon \rfloor) \\
&= \frac{k(r+1)(r+2)}{2} + 1 + 2 + \cdots + r.
\end{aligned}$$

Note that

$$\begin{aligned}
(r+2)\varepsilon &\geq \frac{r(r+2)}{r+1} \\
&= r + \left(1 - \frac{1}{r+1}\right) \\
&= r + \frac{r}{r+1} \\
(r+3)\varepsilon &\geq \frac{r(r+3)}{r+1} \\
&= r + 2 - \frac{2}{r+1} \\
&= r + 1 + \frac{r-1}{r+1}
\end{aligned}$$

holds. Combining the above inequalities with $(r+2)\varepsilon < (r+3)\varepsilon < 1$, it follows that $\lfloor (r+2)\varepsilon \rfloor$ is equal to r or $r+1$, and $\lfloor (r+3)\varepsilon \rfloor$ is equal to $r+1$ or $r+2$. By hypothesis, $r+3$ divides the integer

$$\begin{aligned}
&\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor (r+1)\alpha \rfloor + \lfloor (r+2)\alpha \rfloor + \lfloor (r+3)\alpha \rfloor \\
&= (k + 2k + \cdots + (r+3)k) + (\lfloor \varepsilon \rfloor + \lfloor 2\varepsilon \rfloor + \cdots + \lfloor (r+3)\varepsilon \rfloor) \\
&= \frac{k(r+3)(r+4)}{2} + 1 + 2 + \cdots + r + \lfloor (r+2)\varepsilon \rfloor + \lfloor (r+3)\varepsilon \rfloor \\
&= \frac{k(r+3)(r+4)}{2} + 1 + 2 + \cdots + r + (r+1) + (r+2) \\
&\quad + (\lfloor (r+2)\varepsilon \rfloor - (r+1)) + (\lfloor (r+3)\varepsilon \rfloor - (r+2)),
\end{aligned}$$

which shows that $r+3$ divides the integer

$$(\lfloor (r+2)\varepsilon \rfloor - (r+1)) + (\lfloor (r+3)\varepsilon \rfloor - (r+2)).$$

Each of the two terms in the above sum is equal to 0 or -1. Since $r+3 \geq 5$, it follows that both terms are equal to 0, and thus $\lfloor (r+3)\varepsilon \rfloor = r+2$. This implies that $\varepsilon \geq \frac{r+2}{r+3}$. □

Claim — We have $\varepsilon = 0$.

Proof of the claim. On the contrary, assume that $\varepsilon > 0$. By the first claim, we have $\varepsilon \geq \frac{1}{2}$. By the second claim, we have $\varepsilon \geq \frac{2}{3}$. The third claim implies that for every even positive integer r, the inequality $\varepsilon \geq \frac{r}{r+1}$ implies $\varepsilon \geq \frac{r+2}{r+3}$. By induction, it follows that $\varepsilon \geq \frac{r}{r+1}$ for every even positive integer r. This implies that $\varepsilon \geq 1$. This contradicts the assumption that $\varepsilon < 1$. Therefore, $\varepsilon = 0$. □

It follows that α is an integer. By hypothesis, the integer

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor |\alpha|\, \alpha \rfloor + \lfloor (|\alpha|+1)\alpha \rfloor = \frac{\alpha(|\alpha|+1)(|\alpha|+2)}{2}$$

is a multiple of $|\alpha|+1$, which implies that 2 divides $\alpha(|\alpha|+2)$, which shows that α is even. Conversely, if α is an even integer, then for any positive integer n, the integer

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor = \frac{n(n+1)}{2}\alpha$$

is a multiple of n. Therefore, the real numbers satisfying the given condition are precisely the even integers. ■

References

[Che25] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289