MOPSS

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Mathematics Olympiad

Problem Solving Sessions

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

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Exercise 1.1 (Cono Sur Olympiad 2013 P1, AoPS)**.** Four distinct points are marked in a line. For each point, the sum of the distances from said point to the other three is calculated; getting in total 4 numbers. Decide whether these 4 numbers can be, in some order:

(i) $29, 29, 35, 37$,

(ii) $28, 29, 35, 37$,

(iii) $28, 34, 34, 37$.

Walkthrough —

(a) For each of the four points, express the sum of the distances to the other three points in terms of the distances between consecutive points among the four points.

(b) Show that at least two of the four numbers must be equal.

(c) How does it follow that the first set of numbers is the only one that can be obtained?

Solution 1. Denote the points by A, B, C, D from left to right. Let x, y, z be the distances between consecutive points, that is, $x = AB, y = BC, z = CD$. For P in $\{A, B, C, D\}$, let S_P be the sum of the distances from P to the other three points. Then, we have

$$\begin{aligned} S_A &= AB + AC + AD \\ &= x + (x+y) + (x+y+z) \\ &= 3x + 2y + z, \\ S_B &= AB + BC + BD \\ &= x + y + (y+z) \\ &= x + 2y + z, \\ S_C &= AC + BC + CD \end{aligned}$$

$$
\begin{aligned}
&= (x+y)+y+z \\
&= x+2y+z, \\
S_D &= AD+BD+CD \\
&= (x+y+z)+(y+z)+z \\
&= x+2y+3z.
\end{aligned}
$$

Note that $S_B = S_C$. Thus, at least two of the numbers S_A, S_B, S_C, S_D are equal. This is only the case for the first set of numbers, $29, 29, 35, 37$. Also observe that for $x = 3, y = 11, z = 4$, we have $S_A = 35, S_B = S_C = 29, S_D = 37$. This shows that only the first set of numbers can occur as the sums of the distances from the four points to the other three points in some order. ■

Exercise 1.2 (Cono Sur Olympiad 2025 P4, AoPS)**.** Lucero and Pablo play a game on the board shown in Fig. 1. Lucero plays first, and they take turns. In each turn, a player chooses an unpainted circle from the bottom row and paints it blue, green, or red. This continues for four turns until the entire bottom row is painted.

Then, the rest of the board is painted according to the following rules:

(i) If two adjacent circles in a row are the same color, the circle above and adjacent to them is painted that same color.

(ii) If two adjacent circles in a row are of different colors, the circle above and adjacent to them is painted with the third color.

This procedure is repeated until all the circles on the board are painted. Lucero wins if the single top circle is painted red or green, and Pablo wins if it is painted blue.

Determine who has a winning strategy.

Walkthrough —

(a) Use the congruence classes modulo 3 to represent the colors of the circles. For example, we can represent blue as 0, green as 1, and red as -1 modulo 3.

(b) Let a, b, c, d be the colors of the circles in the bottom row from left to right, respectively, represented as congruence classes modulo 3.

(c) Express the color of the top circle as a function of the colors of the circles in the bottom row.

(d) Determine a winning strategy for Pablo based on the colors of the circles in the bottom row.

Solution 2. We will show that Pablo has a winning strategy.

Let us consider the colors as congruence classes modulo 3 as follows: 0 for blue, 1 for green, and -1 for red. We can represent the colors of the circles in

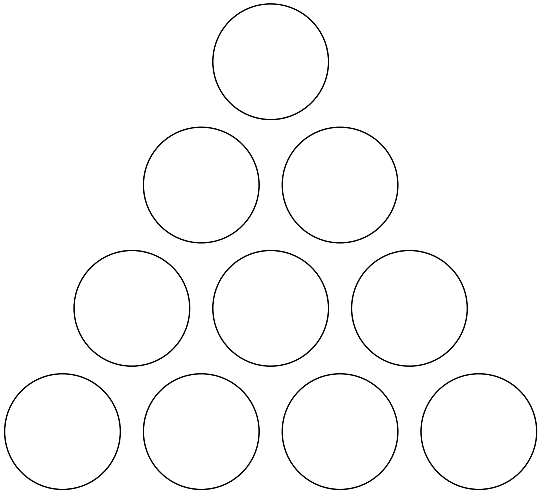

Figure 1: Cono Sur Olympiad 2025 P4, Exercise 1.2

the bottom row, from left to right, as a sequence of four congruence classes, which we can denote as a, b, c, d, where each of a, b, c, d is in $\{0, 1, -1\}$. Note that if the colors of two adjacent circles are x, y modulo 3, then the color of the circle above them is $-(x+y)$ modulo 3. Indeed, if the colors of two adjacent circles are x, y modulo 3, and x, y are congruent modulo 3, then the color of the circle above them is x modulo 3, which is $-(x+y)$ modulo 3 since $x \equiv y \pmod 3$. If x, y are not congruent modulo 3, then the color of the circle above them is the third color, which is $-(x+y)$ modulo 3. Therefore, the color of the top circle can be expressed as a function of a, b, c, d as follows. In fact, note the colors of the circle in the third row are $-(a+b)$, $-(b+c)$, and $-(c+d)$ modulo 3. Then, the colors of the circles in the second row are

$$-(-(a+b) + -(b+c)) = a + 2b + c,$$
$$-(-(b+c) + -(c+d)) = b + 2c + d$$

modulo 3. Finally, the color of the top circle is

$$-(a + 2b + c + b + 2c + d) = -a - 3b - 3c - d \equiv -a - d \pmod 3.$$

Therefore, the color of the top circle is determined by the colors of the leftmost and rightmost circles in the bottom row.

If Lucero, in her first turn, chooses to color the leftmost or the rightmost circle in the bottom row, then Pablo colors the circle on the opposite end of the bottom row with its negative congruence class modulo 3. This ensures that the color of the top circle is 0 modulo 3, which means that the top circle is blue, and Pablo wins.

If Lucero, in her first turn, chooses to color any of the two middle circles in the bottom row, then Pablo colors the other middle circle with any color. In next turn, Lucero colors one of the two remaining circles in the bottom row, and Pablo colors the other remaining circle in the bottom row with its negative congruence class modulo 3. This again ensures that the color of the top circle is 0 modulo 3, which means that the top circle is blue, and Pablo wins.

Therefore, Pablo has a winning strategy, and the answer is Pablo. ■

Exercise 1.3 (Japan Mathematical Olympiad 2018 P1, AoPS)**.** Positive integers between 1 to 100 inclusive are written on a blackboard, each exactly once. One operation involves choosing two numbers a and b on the blackboard and erasing them, then writing the greatest common divisor of a^2+b^2+2 and a^2b^2+3. After a number of operations, there is only one positive integer left on the blackboard. Prove this number cannot be a perfect square.

Walkthrough —

(a) Show that the parity of the number of multiples of 3 on the blackboard is an invariant.

(b) Show that the last number on the blackboard is a multiple of 3, but is not a multiple of 9.

Solution 3.

Claim — The parity of the number of multiples of 3 on the blackboard is an invariant.

Proof of the claim. Let a and b be the two numbers chosen in an operation. If a and b are both multiples of 3, then a^2+b^2+2 is not divisible by 3, and hence, $\gcd(a^2+b^2+2, a^2b^2+3)$ is not divisible by 3. If exactly one of a and b is a multiple of 3, then a^2+b^2+2 and a^2b^2+3 are both divisible by 3, and hence, $\gcd(a^2+b^2+2, a^2b^2+3)$ is divisible by 3. If neither a nor b is a multiple of 3, then a^2+b^2+2 is not divisible by 3, and hence, $\gcd(a^2+b^2+2, a^2b^2+3)$ is not divisible by 3. Therefore, the parity of the number of multiples of 3 on the blackboard remains unchanged after each operation. □

Initially, there are 33 multiples of 3 on the blackboard, so the parity of the number of multiples of 3 is odd. After performing 99 operations, there is only one number left on the blackboard, and it must be a multiple of 3. Let x, y be the last two numbers chosen in the final operation. Then, the last number on the blackboard is $\gcd(x^2+y^2+2, x^2y^2+3)$. This shows that 3 divides x^2y^2+3, which is implies that 3 divides xy. Therefore, 9 does not divide x^2y^2+3, and hence, 9 does not divide $\gcd(x^2+y^2+2, x^2y^2+3)$. This shows that the last number on the blackboard is divisible by 3, but is not divisible by 9, and hence is not a perfect square. ■

Exercise 1.4 (St Petersburg Mathematical Olympiad 2025 Round 2 Grade 11 P1, AoPS)**.** 100 red and 99 blue piranhas were released into the empty Quiet Pool. When a piranha wants to eat, it can eat two other piranhas. If its victims are of the same color, the piranha changes its color, but if they are multicolored, it does not change. In the end, only one piranha remains. What color is it?

Walkthrough —

(a) Show that the number of blue piranhas and the number of red piranhas differ by a multiple of 4.

(b) Show that a red piranha remains at the end of the process.

Solution 4. Let r, b denote the number of red and blue piranhas, respectively, before a piranha eats. If a red piranha eats two red piranhas, then r, b changes to $r-3, b+1$ respectively. If a red piranha eats two blue piranhas, then r, b changes to $r-1, b-1$ respectively. If a blue piranha eats two red piranhas, then r, b changes to $r-1, b-1$ respectively. If a blue piranha eats two blue piranhas, then r, b changes to $r+1, b-3$ respectively. If the victims are of different colors, then r, b changes to $r-1, b-1$ respectively, regardless of the color of the eater. If R (resp. B) denotes the number of red (resp. blue) piranhas at the end of this eating, then the integers $R-B, r-b$ differ by a multiple of 4. This shows that the difference between the number of red and blue piranhas remains the same modulo 4 throughout the process. Since initially there are 100 red and 99 blue piranhas, the difference is $100-99=1$. Hence, the difference between the number of red and blue piranhas at the end is also $1 \pmod 4$. Therefore, the last remaining piranha must be red, since if it were blue, the difference would be -1, which is not congruent to $1 \pmod 4$. ■

Remark. We can ask to determine the colors of the remaining piranhas at the end, if initially there were 100 red and 100 blue piranhas.

Exercise 1.5 (British Mathematical Olympiad Round 2 2023 P2, AoPS)**.** For an integer $n > 1$, the numbers $1, 2, 3, \ldots, n$ are written in order on a blackboard. The following *moves* are possible:

(i) Take three adjacent numbers x, y, z whose sum is a multiple of 3 and replace them with y, z, x.

(ii) Take two adjacent numbers x, y whose difference is a multiple of 3 and replace them with y, x.

For example we could take: $1, 2, 3, 4 \xrightarrow{(i)} 2, 3, 1, 4 \xrightarrow{(ii)} 2, 3, 4, 1$. Find all n such that the initial list can be transformed into $n, 1, 2, \ldots, n-1$ after a finite number of moves.

Walkthrough —

(a)

Solution 5. Let $x_1, x_2, \dots, x_n$ denote the numbers on the blackboard, in the order they are written, prior to a move. We will show that the modulo 3 congruence class of the following quantity is invariant under the moves:

$$S = \sum_{i=1}^{n} i x_i.$$

For move (i), we have $x_i = x$, $x_{i+1} = y$ and $x_{i+2} = z$ for some $1 \leq i \leq n-2$. Then the change in S is

$$\begin{aligned}
&(iy + (i+1)z + (i+2)x) - (ix + (i+1)y + (i+2)z) \\
&= -y - z + 2x \\
&\equiv -(x+y+z) \pmod 3 \\
&\equiv 0 \pmod 3.
\end{aligned}$$

For move (ii), we have $x_i = x$ and $x_{i+1} = y$ for some $1 \leq i \leq n-1$. Then the change in S is

$$iy + (i+1)x - (ix + (i+1)y) = x - y \equiv 0 \pmod 3.$$

Thus S is invariant modulo 3. Initially, we have

$$\begin{aligned}
S &= 1 \cdot 1 + 2 \cdot 2 + \cdots + n \cdot n \\
&= 1^2 + 2^2 + \cdots + n^2.
\end{aligned}$$

Let $n > 1$ be an integer such that the initial list can be transformed into $n, 1, 2, \dots, n-1$ after a finite number of moves. Then we also have

$$\begin{aligned}
S &= 1 \cdot n + 2 \cdot 1 + 3 \cdot 2 + \cdots + n \cdot (n-1) \\
&= n + (2^2 - 2) + (3^2 - 3) + \cdots + (n^2 - n) \\
&\equiv 1^2 + 2^2 + \cdots + n^2 - (1 + 2 + \cdots + n) + n.
\end{aligned}$$

Since S is invariant modulo 3, we have

$$1^2 + 2^2 + \cdots + n^2 \equiv 1^2 + 2^2 + \cdots + n^2 - (1 + 2 + \cdots + n) + n \pmod 3,$$

which simplifies to

$$1 + 2 + \cdots + n \equiv n \pmod 3.$$

This gives

$$\frac{n(n-1)}{2} \equiv 0 \pmod 3,$$

and hence, either $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$.

Claim — Let $n > 1$ be an interger satisfying $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$. Then the initial list can be transformed into $n, 1, 2, \ldots, n-1$ after a finite number of moves.

Proof of the claim. Note that the move (i) can be applied to $1, 2, 3$ to get $2, 3, 1$, and the move (i) can be applied again to $2, 3, 1$ to get $3, 1, 2$. Thus we can transform $1, 2, 3$ into $3, 1, 2$. Similarly, the move (i) can be applied to $1, 2, 3, 4$ to get $1, 3, 4, 2$, and the move (i) can be applied again to $1, 3, 4, 2$ to obtain $1, 4, 2, 3$. Applying the move (ii) to $1, 4, 2, 3$ gives $4, 1, 2, 3$. Thus we can transform $1, 2, 3, 4$ into $4, 1, 2, 3$. Let k be a positive integer such that the list $1, 2, \ldots, n$ can be transformed into $n, 1, 2, \ldots, n-1$ for all $1 < n \leq 3k+1$ such that 3 divides $n(n-1)$.

Consider the list $1, 2, \ldots, 3k+3$. Since $4, 5, \ldots, 3k+3$ are congruent to $1, 2, \ldots, 3k$ modulo 3 respectively, by the induction hypothesis, we can transform $4, 5, \ldots, 3k+3$ into $3k+3, 4, 5, \ldots, 3k+2$. Applying the move (ii) to $1, 2, 3, 3k+3, 4, 5, \ldots, 3k+2$ gives $1, 2, 3k+3, 3, 4, 5, \ldots, 3k+2$. Applying the move (i) to $1, 2, 3k+3$ twice gives $3k+3, 1, 2$. Thus we can transform $1, 2, \ldots, 3k+3$ into $3k+3, 1, 2, 3, 4, \ldots, 3k+2$.

Consider the list $1, 2, \ldots, 3k+4$. Since $4, 5, \ldots, 3k+4$ are congruent to $1, 2, \ldots, 3k+1$ modulo 3 respectively, by the induction hypothesis, we can transform $4, 5, \ldots, 3k+4$ into $3k+4, 4, 5, \ldots, 3k+3$. Applying move (i) to $1, 2, 3, 3k+4$ gives $1, 2, 3k+4, 3$. Applying move (i) to $1, 2, 3k+4$ twice gives $3k+4, 1, 2$. Thus we can transform $1, 2, \ldots, 3k+4$ into $3k+4, 1, 2, 3, 4, \ldots, 3k+3$.

This completes the induction step, and hence the claim holds by induction. □

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References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)