MOPSS

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Mathematics Olympiad

Problem Solving Sessions

DEPARTMENT OF MATHEMATICS
IISER BHOPAL

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [Che25, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

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§1

Exercise 1.1 (Belarus National Olympiad 2023 Grade 8 Day 1 P1, AoPS)**.** A move on an unordered triple of numbers (a, b, c) changes the triple to either $(a, b, 2a + 2b - c)$, $(a, 2a + 2c - b, c)$ or $(2b + 2c - a, b, c)$. Can you perform a finite sequence of moves on the triple $(3, 5, 14)$ to get the triple $(3, 13, 6)$?

> **Walkthrough** — Show that if a sequence of moves is performed on a triple (a, b, c), then the resulting **unordered** triple is congruent to one of the triples
>
> $$(a, b, -(a + b + c)), (a, -(a + b + c), c), (-(a + b + c), b, c), (a, b, c)$$
>
> modulo 3, that is, the residues of the entries of the resulting triple modulo 3 coincide with the residues of the entries of one of the above triples in some order.

Solution 1.

■

Exercise 1.2 (Belarus National Olympiad 2023 Grade 9 Day 1 P2, AoPS)**.** A move on an unordered triple of numbers (a, b, c) changes the triple to either $(a, b, 2a + 2b - c)$, $(a, 2a + 2c - b, c)$ or $(2b + 2c - a, b, c)$. Can you perform a finite sequence of moves on the triple $(3, 5, 14)$ to get the triple $(9, 8, 11)$?

> **Walkthrough** — Show that the mod 4 congruence class of the sum of the entries of such a triple remains invariant under the allowed moves if the sum of the entries of the initial triple is congruent to 2 modulo 4.

Solution 2.

■

Exercise 1.3 (Belarus National Olympiad 2024 Grade 11 Day 2 P6, AoPS, by I. Voronovich)**.** Let $2 = p_1 < p_2 < \cdots < p_n < \dots$ be all prime numbers. Prove that for any positive integer $n \geq 3$ there exist at least $p_n + n - 1$ prime numbers, that do not exceed $p_1 p_2 \dots p_n$.

Walkthrough —

(a) Consider the numbers

$$\begin{aligned}&p_1, p_2, \ldots, p_{n-1},\\ &p_1p_2\ldots p_{n-1} - 1, 2p_1p_2\ldots p_{n-1} - 1,\\ &3p_1p_2\ldots p_{n-1} - 1, \ldots, p_np_1p_2\ldots p_{n-1} - 1.\end{aligned}$$

(b) Show that these are pairwise relatively prime.

Solution 3. Let $n \geq 3$ be an integer. Note that the integers

$$p_1p_2\ldots p_{n-1} - 1, 2p_1p_2\ldots p_{n-1} - 1, 3p_1p_2\ldots p_{n-1} - 1, \ldots, p_np_1p_2\ldots p_{n-1} - 1$$

are pairwise relatively prime. Indeed, if a prime p divides two of them, say $ap_1p_2\ldots p_{n-1} - 1$ and $bp_1p_2\ldots p_{n-1} - 1$ with $1 \leq a < b \leq p_n$, then p divides their difference $(b-a)p_1p_2\ldots p_{n-1}$. Since p cannot be any of the primes $p_1, p_2, \ldots, p_{n-1}$, it follows that p divides $b-a$. Since $1 \leq b-a < p_n$, we conclude that $p < p_n$, that is, p is one of the primes $p_1, p_2, \ldots, p_{n-1}$. However, this is impossible since none of these primes divides $ap_1p_2\ldots p_{n-1} - 1$. Thus, the numbers

$$p_1p_2\ldots p_{n-1} - 1, 2p_1p_2\ldots p_{n-1} - 1, 3p_1p_2\ldots p_{n-1} - 1, \ldots, p_np_1p_2\ldots p_{n-1} - 1$$

are pairwise relatively prime. Since $n \geq 3$, we obtain

$$p_1p_2\ldots p_{n-1} - 1 \geq 2p_{n-1} - 1 > p_{n-1}.$$

This implies that the numbers

$$\begin{aligned}&p_1, p_2, \ldots, p_{n-1},\\ &p_1p_2\ldots p_{n-1} - 1, 2p_1p_2\ldots p_{n-1} - 1,\\ &3p_1p_2\ldots p_{n-1} - 1, \ldots, p_np_1p_2\ldots p_{n-1} - 1,\end{aligned}$$

are all distinct and greater than 1. Moreover, these $p_n + n - 1$ numbers are pairwise relatively prime. Therefore, each of them has at least one prime divisor, which does not divide any of the other numbers. Since all these numbers are less than or equal to $p_1p_2\ldots p_n$, we conclude that there are at least $p_n + n - 1$ distinct prime numbers that do not exceed $p_1p_2\ldots p_n$. ■

Exercise 1.4 (INMO 2026 P2, AoPS)**.** Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function satisfying the following condition: for each $k > 2026$, the number $f(k)$ equals the maximum number of times a number appears in the list $f(1), f(2), \ldots, f(k-1)$. Prove that $f(n) = f(n + f(n))$ for infinitely many $n \in \mathbb{N}$. (Here $\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$ of positive integers.)

Walkthrough —

(a) Show that $f|_{\{2027,2028,\dots\}}$ is a non-decreasing function.

(b) Show that f takes arbitrarily large values.

(c) Show that there exists an integer $r \geq 2027$ such that

$$f(r) > \max\{f(1), f(2), \dots, f(r-1)\}.$$

(d) Show that for such an integer r, the values of f at the integers $r, r+1, r+2, \dots, r+f(r)$ are all equal, and $f(r+f(r)+1) = f(r)+1$.

(e) Conclude the solution.

Solution 4. Note that $f(n) \leq f(n+1)$ holds for any $n \geq 2027$, which shows that $f|_{\{2027,2028,\dots\}}$ is a non-decreasing function. For any $m \geq 2027$, note that $f(m)$ is a positive integer. Observe that for any $m \geq 2027$, if

$$f(m), f(m+1), f(m+2), \dots, f(m+f(m))$$

are all equal, then $f(m+f(m)+1)$ is greater than or equal to $f(m)+1$. This shows that

$$f(m) \geq 1, \quad f(m) < f(m+f(m)+1)$$

hold for any $m \geq 2027$. This shows that f takes arbitrarily large values.

Claim — Let $r \geq 2027$ be an integer such that

$$f(r) > \max\{f(1), f(2), \dots, f(r-1)\}.$$

Then f takes equal values at the integers

$$r, r+1, r+2, \dots, r+f(r)$$

and $f(r+f(r)+1) = f(r)+1$.

Proof of the Claim. The maximum number of times that a number appears in the list $f(1), f(2), \dots, f(r-1)$ is equal to $f(r)$. Using the inequality $f(r) > \max\{f(1), f(2), \dots, f(r-1)\}$, we see that any such number is less than $f(r)$. Therefore, the numbers $f(r), f(r+1), \dots, f(r+f(r))$ are equal. It follows that $f(r+f(r)+1) = f(r)+1$. □

Since f takes arbitrarily large values, there exists an integer $r \geq 2027$ such that $f(r) > \max\{f(1), f(2), \dots, f(r-1)\}$. Using the Claim and applying induction, it follows that the images of $r, r+1, r+2, \dots$ under f are equal to

$$\underbrace{t, t, \dots, t}_{t+1 \text{ times}}, \underbrace{t+1, t+1, \dots, t+1}_{t+2 \text{ times}}, \underbrace{t+2, t+2, \dots, t+2}_{t+3 \text{ times}}, \dots$$

respectively, where $t = f(r)$. This implies that $f(n) = f(n + f(n))$ for infinitely many $n \in \mathbb{N}$, as desired. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)