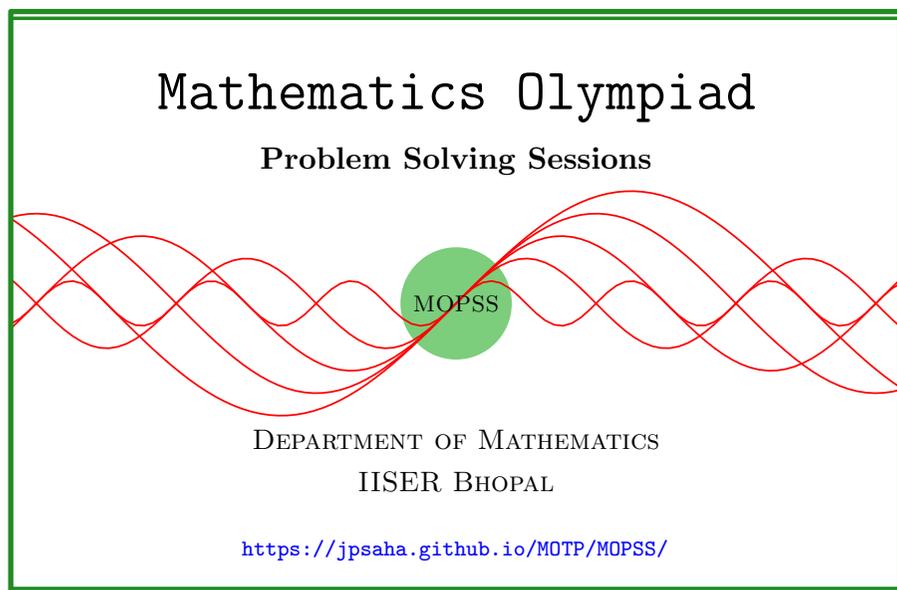


MOPSS

21 February 2026

The logo is enclosed in a green rectangular border. At the top, the text "Mathematics Olympiad" is written in a large, black, serif font. Below it, "Problem Solving Sessions" is written in a smaller, black, serif font. In the center, there is a green circle containing the text "MOPSS" in white. This circle is overlaid by several red, wavy lines that resemble mathematical sine waves. Below the waves, the text "DEPARTMENT OF MATHEMATICS" and "IISER BHOPAL" is written in a black, serif font. At the bottom, the URL "https://jpsaha.github.io/MOTP/MOPSS/" is written in a blue, sans-serif font.

Mathematics Olympiad

Problem Solving Sessions

MOPSS

DEPARTMENT OF MATHEMATICS
IISER BHOPAL

<https://jpsaha.github.io/MOTP/MOPSS/>

Suggested readings

- Evan Chen's advice [On reading solutions](https://blog.evanchen.cc/2017/03/06/on-reading-solutions/), available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
- Evan Chen's [Advice for writing proofs/Remarks on English](https://web.evanchen.cc/handouts/english/english.pdf), available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- [Notes on proofs](#) by Evan Chen from [OTIS Excerpts](#) [[Che25](#), Chapter 1].
- [Tips for writing up solutions](https://www.math.utoronto.ca/barbeau/writingup.pdf) by Edward Barbeau, available at <https://www.math.utoronto.ca/barbeau/writingup.pdf>.
- Evan Chen discusses why [math olympiads are a valuable experience for high schoolers](#) in the post on [Lessons from math olympiads](https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/), available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

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§1

Exercise 1.1 (Austrian Mathematical Olympiad, National Competition, Preliminary Round 2025 P3, AoPS, proposed by Michael Reitmeir). Consider the following game for a positive integer n . Initially, the numbers $1, 2, \dots, n$ are written on a board. In each move, two numbers are selected such that their difference is also present on the board. This difference is then erased from the board. (For example, if the numbers 3, 6, 11 and 17 are on the board, then 3 can be erased as $6 - 3 = 3$, or 6 as $17 - 11 = 6$, or 11 as $17 - 6 = 11$.)

For which values of n , is it possible to end with only one number remaining on the board?

Walkthrough — Assume that it is possible to end with only one number remaining on the board.

- (a) Show that if d is an odd integer dividing the integers present on the board at some stage, then d divides the integer that was erased in the move prior to this stage, that is, d divides all the integers present on the board prior to this stage as well.
- (b) Show that n has no odd positive divisor other than 1.

Next, assume that n is a power of 2. Use induction to show that it is possible to end with only one number remaining on the board.

Solution 1. By the claims below, it follows that it is possible to end with only one number remaining on the board if and only if n is a power of 2.

Claim — Let n be a positive integer. Assume that it is possible to end with only one number remaining on the board. Then n is a power of 2.

Proof of the Claim. Let d be an odd positive divisor of n . We will show that d must be 1. Since no two of the numbers $1, 2, \dots, n$ differ by n , the number n cannot be erased in any move. Therefore, n must be the last remaining number on the board.

Assume that d divides all the integers present on the board at some stage in the game. Denote these integers by a_1, a_2, \dots, a_k . Denote by b the integer that was erased in the move prior to this stage.

Consider the case that b is the difference of b and a_i for some i . Then b is equal to one of $b - a_i, a_i - b$. Since a_i is positive, we obtain $b = a_i - b$, which implies $a_i = 2b$. Since d divides a_i and d is odd, it follows that d divides b .

Now, consider the case that b is equal to $a_i - a_j$ for some i and j with $i \neq j$. Since d divides both a_i and a_j , it follows that d divides b .

This proves that if d divides all the integers present on the board at some stage, then d divides the integer that was erased in the move prior to this stage, that is, d divides all the integers present on the board prior to this stage as well. Since d divides n , it follows that d divides all the integers present on the board at every stage prior to the last stage. In particular, d divides 1, which implies $d = 1$.

This proves that n has no odd positive divisor other than 1. Therefore, n is a power of 2. □

Claim — Let n be a positive integer. Assume that n is a power of 2. Then it is possible to end with only one number remaining on the board.

Proof of the Claim. Write $n = 2^k$ with $k \geq 0$. Using induction, we will prove the claim. For the base case $k = 0$, we have $n = 1$. In this case, we already have only one number remaining on the board, so the claim holds. Also note that the claim holds for $k = 1$ as well, since we can erase 1 and be left with 2.

Assume that the claim holds for some $k \geq 1$, that is, for $n = 2^k$. Consider $n = 2^{k+1}$. We can divide the numbers $1, 2, \dots, 2^{k+1}$ into two groups: $1, 2, \dots, 2^k$ and $2^k + 1, 2^k + 2, \dots, 2^{k+1}$. Note that

$$2^k + i = 2^{k+1} - (2^k - i)$$

for any i with $1 \leq i \leq 2^k - 1$. Therefore, we can erase the numbers $2^k + 1, 2^k + 2, \dots, 2^{k+1} - 1$ by using the numbers $1, 2, \dots, 2^k - 1$ and 2^{k+1} . Thus, we can reduce the numbers on the board to $1, 2, \dots, 2^k$ and 2^{k+1} . By the induction hypothesis, we can reduce $1, 2, \dots, 2^k$ to 2^k . Finally, when we are left with 2^k and 2^{k+1} on the board, we can erase 2^k and be left with 2^{k+1} as the only number remaining on the board. By induction, the claim holds for all $k \geq 0$. □

■

Exercise 1.2 (Dutch Mathematical Olympiad 2025 P3, AoPS). Amber has a square consisting of 10×10 square boxes. One at a time, she randomly chooses one of the hundred squares and fills it with a number. The number she puts in a square is always equal to the total number of squares already filled in that square's row and column. So in the first square she chooses she writes a 0, in the second square a 0 or a 1, depending on whether it is in a different row and a different column or not, et cetera. So in each square there will be a number from 0 to 18 inclusive. After Amber has written a number in each square, she adds up all the numbers. What is/are the possible result(s) of this addition?

Walkthrough —

- (a) In a chosen square, instead of writing the sum of the number of filled squares in the row and column, let us write down the pair of numbers (r, c) where r is the number of filled squares in that row before filling the square, and c is the number of filled squares in that column before filling the square.
- (b) It amounts to find the sum of the numbers of the form $r + c$ across all the squares.
- (c) Try to sum up the r values across all the squares, and then sum up the c values across all the squares.
- (d) Show that the result of the addition is always the same, regardless of the order in which Amber fills the squares.

Solution 2. Let us allow Bob to play along with Amber. As Amber fills a square, Bob fills the same square with a pair (r, c) where r is the number of filled squares in that row before Amber fills it, and c is the number of filled squares in that column before Amber fills it. Note that the number Amber fills in that square is $r + c$. After Amber fills all the squares, Bob will have filled all the squares with pairs of numbers. Let us denote the sum of all the numbers Amber filled in as S . We have

$$S = \sum_{(r,c)} (r + c) = \sum_{(r,c)} r + \sum_{(r,c)} c.$$

Now, let us look at the first sum $\sum_{(r,c)} r$. For each row, the value of r across the squares in that row is $0, 1, 2, \dots, 9$ in some order. Therefore, the sum of r across the squares in that row is $0 + 1 + 2 + \dots + 9 = 45$. Since there are 10 rows, we have

$$\sum_{(r,c)} r = 10 \cdot 45 = 450.$$

By a similar argument, we have

$$\sum_{(r,c)} c = 10 \cdot 45 = 450.$$

Therefore, we have

$$S = \sum_{(r,c)} r + \sum_{(r,c)} c = 450 + 450 = 900.$$

Thus, the only possible result of the addition is 900. ■

Exercise 1.3 (Columbia Junior Mathematical Olympiad 2024 P3, AoPS, proposed by Santiago Rodriguez). Let $n > 1$ be a positive integer. There are n rocks arranged in a circle, with one frog sitting on each rock. The numbers $1, 2, \dots, n$ are written on the rocks in some order. After a signal is given, each frog looks at the number i written on the rock it occupies and then jumps i positions clockwise along the circle. Determine all values of n for which it is possible to assign the numbers $1, 2, \dots, n$ to the rocks in such a way that, after the signal, no two frogs land on the same rock.

Walkthrough —

- (a) Label the rocks as the first, second, \dots , n -th rock in the clockwise order. For each $i \in \{1, 2, \dots, n\}$, let a_i be the number written on the i -th rock.
- (b) Observe that if n is a positive integer such that there is an assignment of the numbers $1, 2, \dots, n$ to the rocks such that no two frogs land on the same rock after the signal, then the numbers $i + a_i$ for $i = 1, 2, \dots, n$ are distinct modulo n .
- (c) Show that integer n satisfying the given condition must be odd.
- (d) Show that if n is odd, then there is an assignment of the numbers $1, 2, \dots, n$ to the rocks such that no two frogs land on the same rock after the signal.

Solution 3. Let $n > 1$ be a positive integer such that there is an assignment of the numbers $1, 2, \dots, n$ to the rocks such that no two frogs land on the same rock after the signal. Label the rocks as the first, second, \dots , n -th rock in the clockwise order. For each $i \in \{1, 2, \dots, n\}$, let a_i be the number written on the i -th rock. Then, after the signal, the frog on the i -th rock jumps to the $(i + a_i)$ -th rock (where the integer $i + a_i$ is taken modulo n). Since no two frogs land on the same rock, the numbers $i + a_i$ for $i = 1, 2, \dots, n$ are distinct modulo n . In particular, the numbers $i + a_i$ for $i = 1, 2, \dots, n$ are congruent to the numbers $1, 2, \dots, n$ in some order modulo n . Therefore, we have

$$(1 + a_1) + (2 + a_2) + \dots + (n + a_n) \equiv 1 + 2 + \dots + n \pmod{n},$$

which implies that $a_1 + a_2 + \dots + a_n \equiv 0 \pmod{n}$. Since $a_1 + a_2 + \dots + a_n = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$, we have $\frac{n(n+1)}{2} \equiv 0 \pmod{n}$. This implies that $n + 1$ is even, so n is odd.

Conversely, let $n > 1$ be an odd positive integer. Consider the assignment of the numbers $1, 2, \dots, n$ to the rocks in the order $1, 2, \dots, n$ in the clockwise direction. Then, after the signal, the frog on the i -th rock jumps to the $(i+i)$ -th rock, which is the $(2i)$ -th rock modulo n . Since n is odd, the numbers $2i$ for $i = 1, 2, \dots, n$ are distinct modulo n . Therefore, no two frogs land on the same rock after the signal.

Hence, the integers $n > 1$ for which it is possible to assign the numbers $1, 2, \dots, n$ to the rocks in such a way that no two frogs land on the same rock after the signal are precisely the odd integers. ■

Exercise 1.4 (Columbia National Olympiad 2024 P1, AoPS, proposed by Juan David Restrepo). Daniel and Juan David have three piles of stones consisting of 30 yellow stones, 20 blue stones and 10 red stones respectively. They then take turns to remove one stone from any pile, starting with Daniel. Daniel wins if at any moment there are two piles with the same positive number of stones; in any other case, Juan David wins. Determine which of the two players has a winning strategy and describe it.

Walkthrough —

- (a) Show that if Daniel removes a yellow stone during his turns, then he will win.
- (b) Assume that after k turns in total, only one yellow stone is left.
- (c) Assume that among these k turns, Daniel has removed x yellow stones, and Juan David has removed y yellow stones, b blue stones and r red stones.
- (d) Show that $b + r \leq b + r + y \leq x \leq x + y = 29$.
- (e) Prove that the difference between the number of yellow stones and the number of blue stones or the difference between the number of yellow stones and the number of red stones is less than or equal to zero after k turns.
- (f) Conclude that Daniel wins.

Solution 4. The first player, Daniel has a winning strategy. The strategy of Daniel is to remove a yellow stone during his turns. We claim that Daniel will win by following this strategy. More precisely, we will show that Daniel will win when only one yellow stone is left or earlier. Assume that after k turns in total, only one yellow stone is left. Assume that among these k turns, Daniel has removed x yellow stones, and Juan David has removed y yellow stones, b blue stones and r red stones. Note that after these k turns, there are $30 - x - y = 1$ yellow stones, $20 - b$ blue stones and $10 - r$ red stones. Since Daniel is the first player, it follows that at any stage of the game, the number of turns of Juan David is either equal to or one less than the number of turns of Daniel. This implies that $b + r \leq b + r + y \leq x \leq x + y = 29$. This shows

that at least one of the inequalities $b \leq 19$, $r \leq 9$ holds. This gives that after k turns, at least one of the differences

$$\begin{aligned} &\text{the number of yellow stones} - \text{the number of blue stones} = b - 19, \\ &\text{the number of yellow stones} - \text{the number of red stones} = r - 9 \end{aligned}$$

is less than or equal to zero. Observe that these two differences were both positive at the beginning of the game. Therefore, at some moment during the game, within these k turns, one of these two differences must have become zero. In other words, at some moment during the game, one of the red and blue piles had the same positive number of stones as the yellow pile. Noting that during these k turns, the number of yellow stones is always positive, we conclude that Daniel wins. ■

Exercise 1.5 (Brazil National Olympiad 2025 Level 2 P2, AoPS). Let m be a positive integer. Ana and Banana play the following game on a 5×5 board, which is initially filled with 0's in all squares. Taking turns starting with Ana, they choose a square on the board and add an integer from the set $\{1, 2, 3, 4, 5\}$ to the number currently in that square, such that the number in the square does not exceed m . The winner is the player who makes a move that completes a row, a column, or one of the two main diagonals with the number m in all five squares. For which positive integers m does Ana, the first player, have a winning strategy?

Walkthrough —

(a)

Solution 5.

Claim — If m is a multiple of 6, then Banana has a winning strategy.

Proof of the claim.

□

Claim — If m is not a multiple of 6, then Ana has a winning strategy.

Proof of the claim.

□

■

Exercise 1.6 (Belarus National Olympiad 2022 Grade 9 Day 1 P4, AoPS). The numbers $1, 2, \dots, 50$ are written on the board. Anya does the following operation: removes the numbers a and b from the board and writes their sum $a + b$, after which she also notes down the number $ab(a + b)$. After 49 of these operations, only one number is left on the board. Anya summed up all the 49 numbers in her notes and got S .

- (a) Prove that S does not depend on the order of Anya's actions.
 (b) Calculate S .

Walkthrough —

- (a) Let us denote the sum of the cubes of the numbers on the board at a moment as B , and the sum of the numbers in Anya's notes at the same moment as N .
 (b) Prove that $B - 3N$ is invariant under Anya's operation.
 (c) Calculate S using the invariance of $B - 3N$.

Solution 6. Let B denote the sum of the cubes of the numbers on the board at a moment, and N denote the sum of the numbers in Anya's notes at the same moment. We will show that $B - 3N$ is invariant under Anya's operation. Let a and b be the numbers removed from the board. Note that B changes to $B - a^3 - b^3 + (a + b)^3$ and N changes to $N + ab(a + b)$. This shows that $B - 3N$ changes to

$$B - a^3 - b^3 + (a + b)^3 - 3(N + ab(a + b)) = B - 3N.$$

Since $B - 3N$ is invariant, we have

$$1^3 + 2^3 + \dots + 50^3 = (1 + 2 + \dots + 50)^3 - 3S,$$

which gives

$$\begin{aligned} S &= \frac{(1 + 2 + \dots + 50)^3 - (1^3 + 2^3 + \dots + 50^3)}{3} \\ &= \frac{(1 + 2 + \dots + 50)^3 - (1 + 2 + \dots + 50)^2}{3} \\ &= \frac{1}{3} \left(\left(\frac{50 \cdot 51}{2} \right)^2 \cdot \left(\frac{50 \cdot 51}{2} - 1 \right) \right) \\ &= \frac{1275^2 \cdot 1274}{3}. \end{aligned}$$

■

Exercise 1.7 (Brazil National Olympiad 2022 Level 3 P1, AoPS). A single player game has the following rules: initially, there are 10 piles of stones with $1, 2, \dots, 10$ stones, respectively. A movement consists of making one of the following operations:

- (i) to choose 2 piles, both of them with at least 2 stones, combine them and then add 2 stones to the new pile;

- (ii) to choose a pile with at least 4 stones, remove 2 stones from it, and then split it into two piles with amount of stones in those two piles to be chosen by the player.

The game continues until it is not possible to make an operation. Show that the number of piles with one stone in the end of the game is always the same, no matter how the movements are made.

Walkthrough —

- Let p denote the number of piles and s denote the total number of stones at some stage of the game.
- Note that after applying operation (i), (p, s) changes to $(p - 1, s + 2)$, and after applying operation (ii), (p, s) changes to $(p + 1, s - 2)$.
- Construct a polynomial in p and s using the above observations, which remains invariant under both the operations.

Solution 7. Let p denote the number of piles and s denote the total number of stones at some stage of the game. Note that after applying operation (i), we have $p - 1$ piles and $s + 2$ stones. Moreover, after applying operation (ii), we have $p + 1$ piles and $s - 2$ stones. This shows that $2p + s$ remains invariant under both the operations. Let P denote the number of piles when the game has come to an end by performing the operations in a certain sequence. Note that there are at least $P - 1$ piles containing only one stone. Let S denote the number of stones in the remaining pile. Observe that $1 \leq S \leq 3$. The total number of stones is equal to $S + (P - 1)$. This yields

$$2P + S + P - 1 = 2 \times 10 + (1 + 2 + \cdots + 10),$$

which implies

$$3P + S = 2 \times 10 + (1 + 2 + \cdots + 10) + 1 = 20 + 55 + 1 = 76.$$

This implies that $P = 25$ and $S = 1$, so there are 25 piles with one stone in the end of the game. ■

Remark. Note that applying the second operation to the pile obtained by applying the first operation, may yield the piles, to which the first operation was applied. Thus the game need not terminate. This problem seems to ask that the number of piles with one stone in the end of the game (when an end state is reached) is always the same, no matter how the movements are made.

References

- [Che25] EVAN CHEN. *The OTIS Excerpts*. Available at <https://web.evanchen.cc/excerpts.html>. 2025, pp. vi+289 (cited p. 1)