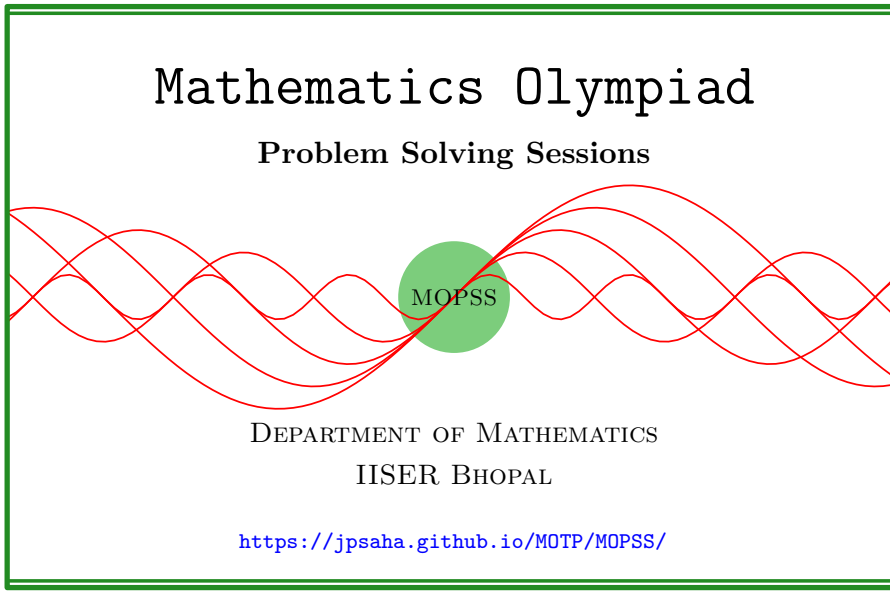
MOPSS

14 February 2026



Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

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§1

Exercise 1.1 (Belarus National Olympiad 2023 Grade 8 Day 1 P1, AoPS)**.** A move on an unordered triple of numbers (a, b, c) changes the triple to either $(a, b, 2a + 2b - c)$, $(a, 2a + 2c - b, c)$ or $(2b + 2c - a, b, c)$. Can you perform a finite sequence of moves on the triple $(3, 5, 14)$ to get the triple $(3, 13, 6)$?

> **Walkthrough** — Show that if a sequence of moves is performed on a triple (a, b, c), then the resulting **unordered** triple is congruent to one of the triples
>
> $$(a, b, -(a + b + c)), (a, -(a + b + c), c), (-(a + b + c), b, c), (a, b, c)$$
>
> modulo 3, that is, the residues of the entries of the resulting triple modulo 3 coincide with the residues of the entries of one of the above triples in some order.

Solution 1.

■

Exercise 1.2 (Belarus National Olympiad 2023 Grade 9 Day 1 P2, AoPS)**.** A move on an unordered triple of numbers (a, b, c) changes the triple to either $(a, b, 2a + 2b - c)$, $(a, 2a + 2c - b, c)$ or $(2b + 2c - a, b, c)$. Can you perform a finite sequence of moves on the triple $(3, 5, 14)$ to get the triple $(9, 8, 11)$?

> **Walkthrough** — Show that the mod 4 congruence class of the sum of the entries of such a triple remains invariant under the allowed moves if the sum of the entries of the initial triple is congruent to 2 modulo 4.

Solution 2.

■

Exercise 1.3 (Columbia National Olympiad 2025 P2, AoPS, proposed by Santiago Rodriguez)**.** The numbers $1, 2, \ldots, 2025$ are written around a circle in some order. On each turn, Celeste chooses an integer $1 \leq n \leq 2025$. Then she selects the number n and the next $n-1$ numbers following it in the clockwise direction and inverts their order on the circle. Prove that, after a finite amount of turns, Celeste can put the numbers on the circle in the order $1, 2, 3, \ldots, 2025$ in the clockwise direction, regardless of their original arrangement on the circle.

Walkthrough —

(a)

Solution 3. Let us prove a few claims about the operation.

Claim — If the numbers $2, 3, x$ appear consecutively in that order in the clockwise direction, then after a finite number of turns, they can be arranged in the order $x, 2, 3$ in the clockwise direction, while keeping the order of the other numbers unchanged.

Proof of the Claim. We can apply the operation with $n = 2$ to the three numbers $2, 3, x$ to get the arrangement $3, 2, x$. Then we apply the operation with $n = 3$ to the three numbers $3, 2, x$ to get the arrangement $x, 2, 3$. Note that the order of the other numbers is unchanged in both operations. □

Claim — Let a, b, c be three distinct integers among $1, 2, \ldots, 2025$ different from $2, 3$. If the numbers $2, 3, a, b, c$ appear consecutively in that order in the clockwise direction, then after a finite number of turns, they can be arranged in the order $2, 3, b, a, c$ in the clockwise direction, while keeping the order of the other numbers unchanged.

Proof of the Claim. We can apply the operation with $n = 3$ to the arrangement $2, 3, a, b, c$ to get the arrangement $2, b, a, 3, c$. Then we apply the operation with $n = 2$ to the arrangement $2, b, a, 3, c$ three times to get the arrangement $b, a, 3, 2, c$. Next, we apply the operation with $n = 3$ to the arrangement $b, a, 3, 2, c$ to get the arrangement $b, a, c, 2, 3$. Note that the order of the other numbers is unchanged in all operations. Applying the prior claim 2020 times, we get the arrangement $2, 3, b, a, c$, without changing the order of the other numbers. □

Claim — Let a, b, c be three distinct integers among $1, 2, \ldots, 2025$ different from $2, 3$. If the numbers a, b, c appear consecutively in this order in the clockwise direction, then after a finite number of turns, they can be

arranged in the order b, a, c in the clockwise direction, while keeping the order of the other numbers unchanged.

Proof of the Claim. Applying the operation with $n = 2$ to 2 and the integer next to 2 in the clockwise direction, we obtain $2, 3$ in this order in the clockwise direction. Then we can apply the previous claim to $2, 3, a, b, c$ to get the arrangement $2, 3, b, a, c$. Next, we apply the first claim to $2, 3$ and the integer next to 3 in the clockwise direction to place $2, 3$ in this order in the clockwise direction so that 3 is placed in the same position as before. Finally, we move 2 back to its original position by applying the operation with $n = 2$ to 2 and the integer next to 2 in the clockwise direction. Note that the order of the other numbers is unchanged in all operations. □

Now, given any arrangement of the numbers $1, 2, \ldots, 2025$ around the circle, we can arrange $2, 3$ in this order in the clockwise direction. Next, applying the previous claim repeatedly, we obtain the arrangement $1, 2, 3$ in this order in the clockwise direction. Finally, we can place the remaining numbers $4, 5, \ldots, 2025$ in the correct order by repeatedly applying the previous claim to adjacent triplets of numbers that are out of order. ■

Here is an alternative solution to the problem.

Solution 4. Let us prove the following claims.

Claim — Let a, b be two distinct integers among $1, 2, \ldots, 2025$ different from $2, 3$. If the numbers $2, 3, x$ appear consecutively in that order in the clockwise direction, then after a finite number of turns, they can be arranged in the order $x, 2, 3$ in the clockwise direction, while keeping the order of the other numbers unchanged.

Proof of the Claim. We can apply the operation with $n = 2$ to the three numbers $2, 3, x$ to get the arrangement $3, 2, x$. Then we apply the operation with $n = 3$ to the three numbers $3, 2, x$ to get the arrangement $x, 2, 3$. Note that the order of the other numbers is unchanged in both operations. □

Claim — If the numbers a, b appear consecutively in that order in the clockwise direction, then after a finite number of turns, they can be arranged in the order b, a in the clockwise direction, while keeping the order of the other numbers unchanged.

Proof of the Claim. Applying the operation with $n = 2$ to 2 and the integer next to 2 in the clockwise direction, we can place $2, 3$ in this order in the clockwise direction. Then applying the prior claim to $2, 3$ and the integer next to 3 in the clockwise direction, we can obtain the arrangement $2, 3, a$ in this order in the clockwise direction. Next, applying the operation with $n = 3$

to the arrangement $2, 3, a, b$ gives the arrangement $2, b, a, 3$. Then we apply the operation with $n = 2$ to the arrangement $2, b, a, 3$ two times to get the arrangement $b, a, 2, 3$. Applying the prior claim to $2, 3$ and the integer next to 3 in the clockwise direction, we can place $2, 3$ in this order in the clockwise direction so that 3 is placed in the same position as before. Finally, we move 2 back to its original position by applying the operation with $n = 2$ to 2 and the integer next to 2 in the clockwise direction. Note that the order of the other numbers is unchanged in all operations. □

Now, given any arrangement of the numbers $1, 2, \ldots, 2025$ around the circle, we can arrange $2, 3$ in this order in the clockwise direction. Next, applying the previous claim a finite number of times, to integers lying between 3 and 4 in the clockwise direction, we can obtain the arrangement $2, 3, 4$ in this order in the clockwise direction. Next, after obtaining the arrangement $2, 3, 4, \ldots, k$ in this order in the clockwise direction, for some $4 \leq k < 2025$, we can apply the previous claim a finite number of times to integers lying between k and $k + 1$ in the clockwise direction to get the arrangement $2, 3, 4, \ldots, k, k + 1$ in this order in the clockwise direction. It follows that we can obtain the arrangement $2, 3, 4, \ldots, 2025$ in this order in the clockwise direction. This gives the desired arrangement $1, 2, 3, \ldots, 2025$ in this order in the clockwise direction. ■

Exercise 1.4 (Kosovo National Olympiad 2023 Grade 11 P1, AoPS)**.** In three different piles, there are 51, 49 and 5 stones, respectively. You can combine two piles in a larger pile or you can separate a pile that contains an even amount of stones into two equal piles. Is it possible that after some turns, we can separate the stones into 105 piles with 1 stone on each pile?

Walkthrough —

(a) Show that the greatest common divisor of the numbers of stones of the piles is an odd integer larger than 1.

(b) Show that it is impossible to separate the stones into 105 piles with 1 stone on each pile.

Solution 5. Note that if the greatest common divisor of the numbers of stones of the piles before some turn is an odd integer d, then d divides the number of stones in each pile after that turn. Since $5, 49, 51$ are all odd, it follows that during the first turn two piles are to be combined. Hence, the numbers of stones in the piles after the first turn are equal to $5, 100$, or to $49, 56$, or to $51, 54$. This shows that the greatest common divisor of the numbers of stones of the piles is an odd integer larger than 1. Consequently, it is impossible to separate the stones into 105 piles with 1 stone on each pile after some turns. ■

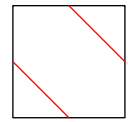


Figure 1: BxMO 2020 P2, Exercise 1.5

Exercise 1.5 (Benelux Mathematical Olympiad 2020 P2, AoPS)**.** Let N be a positive integer. A collection of $4N^2$ unit tiles with two segments drawn on them as shown is assembled into a $2N \times 2N$ board. Tiles can be rotated.

The segments on the tiles define paths on the board. Determine the least possible number and the largest possible number of such paths.

For example, there are 9 *paths on the* 4×4 *board shown below.*

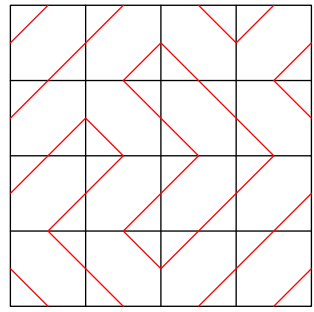


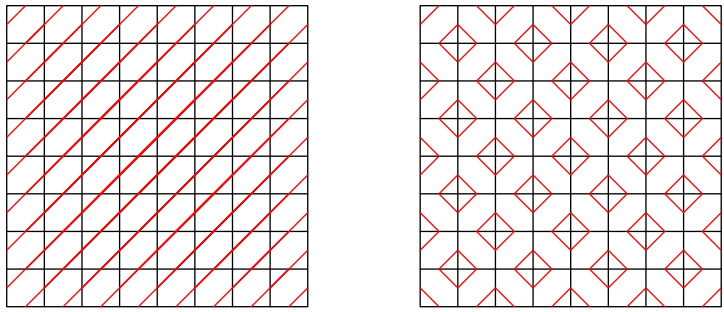
Figure 2: BxMO 2020 P2, Exercise 1.5

Walkthrough —

(a)

Solution 6. Consider a given tiling of a $2N \times 2N$ board using tiles of the given type. Note that there are two types of paths. Some path are between two points lying on the boundary of the square, and the other paths are contained in the interior of the square, and form cycles. Let $\mathcal{B}, \mathcal{C}$ denote the collection of paths of the first and second kind respectively. Observe that among the mid-points of the sides of these $4N^2$ tiles, there are precisely $4 \times 2N = 8N$ mid-points which lie on the boundary of the square. Since each of the paths, which are not cycles, have precisely two such points as end-points, and no two paths of these kind have an end-point in common, it follows that there are precisely $4N$ paths, which are not cycles.

Observe that no two paths have a segment in common. Note that the remaining paths, which are contained in the interior of the square, form cycles. Each of these cycles contains at least four segments. Hence, these cycles



(a) (b)

Figure 3: BxMO 2020 P2, Exercise 1.5

together contain at least $4|\mathcal{C}|$ segments. Also note that a path contains at least two segments, unless it is formed by single segment, and note that there are at least four such paths. Let s denote the number of paths consisting of a single segment. Since there are a total of $4N^2$ tiles, each contributing two segments, we have $8N^2$ segments in total. This shows that

$$8N^2 \geq s + 2(|\mathcal{B}| - s) + 4|\mathcal{C}|,$$

which yields

$$8N^2 + 8N \geq 4(|\mathcal{B}| + |\mathcal{C}|) - s.$$

It follows that

$$|\mathcal{B}| + |\mathcal{C}| \leq 2N^2 + 2N + \frac{s}{4} \leq 2N^2 + 2N + 1.$$

This proves that there are exactly $4N$ paths which are not cycles, and there are at most $2N^2 + 2N + 1$ paths in total.

The tiling, as illustrated in Figure 3a in the case $N = 4$, shows an example of a configuration where the number of paths is equal to $4N$. The tiling, as illustrated in Figure 3b in the case $N = 4$, shows an example of a configuration where the number of paths is equal to $2N^2 + 2N + 1$.

Hence, the least possible number and the maximum possible number of paths are $4N$ and $2N^2 + 2N + 1$ respectively. ■

Exercise 1.6 (Junior Balkan MO 2006 P3, AoPS)**.** We call a number *perfect* if the sum of its positive integer divisors (including 1 and n) equals $2n$. Determine all perfect numbers n for which $n - 1$ and $n + 1$ are prime numbers.

Walkthrough —

(a) Note that 6 is a such a perfect number.

(b) Assume that $n > 6$ is a perfect number such that $n - 1$ and $n + 1$ are

prime numbers.

(c) Show that n is divisible by 6.

(d) Prove that the sum of the positive integer divisors of n is greater than $2n$.

Solution 7. Note that 6 is a such a perfect number. Let $n > 6$ be a perfect number such that $n-1$ and $n+1$ are prime numbers. Since $n-1$ is a prime number and $n-1 \geq 3$, it follows that n is even. Also note that the integers $n-1$ and $n+1$ are larger than 3. Since these are primes, they are not divisible by 3. This shows that n is divisible by 3. Thus, n is divisible by 6. Write $n = 6m$ for some positive integer $m \geq 2$. Then, the sum of the positive integer divisors of n is

$$\geq 1 + m + 2m + 3m + 6m > 12m = 2n,$$

a contradiction. This proves that 6 is the only perfect number such that $n-1$ and $n+1$ are prime numbers. ■

Exercise 1.7 (Kosovo National Olympiad 2023 Grade 12 P4, AoPS)**.** Let p be an odd prime number. Prove that for every such p, regardless of how we divide the set

$$\mathbb{A} = \{1^{p^2-p}, 2^{p^2-p}, \ldots, (p-1)^{p^2-p}, p^2-p+1\},$$

into two disjoint sets, the arithmetic means of the elements of the two subsets cannot be equal.

Walkthrough —

(a)

Solution 8. Let S and T be the two disjoint subsets of $\mathbb{A}$ such that $S \cup T = \mathbb{A}$. Write $S = \{a_1, a_2, \ldots, a_k\}$ and $T = \{b_1, b_2, \ldots, b_{p-k}\}$, where $1 \leq k \leq p-1$. We want to show that the arithmetic means of the elements of S and T are not equal. On the contrary, suppose that they are equal, that is,

$$\frac{a_1 + a_2 + \cdots + a_k}{k} = \frac{b_1 + b_2 + \cdots + b_{p-k}}{p-k},$$

which yields

$$(p-k)(a_1 + a_2 + \cdots + a_k) = k(b_1 + b_2 + \cdots + b_{p-k}).$$

This gives

$$p(a_1 + a_2 + \cdots + a_k)$$

$$= k(a_1 + a_2 + \cdots + a_k + b_1 + b_2 + \cdots + b_{p-k})$$
$$= k(1^{p^2-p} + 2^{p^2-p} + \cdots + (p-1)^{p^2-p} + (p^2 - p + 1)).$$

Since p is a prime, using Euler's theorem, we have $i^{p^2-p} \equiv 1 \pmod{p^2}$ for all $1 \leq i \leq p-1$, which gives

$$1^{p^2-p} + 2^{p^2-p} + \cdots + (p-1)^{p^2-p} + (p^2 - p + 1) \equiv 0 \pmod{p^2}.$$

This shows that p divides $a_1 + a_2 + \cdots + a_k$. Note that the elements of S are congruent to 1 modulo p. This implies $a_1 + a_2 + \cdots + a_k \equiv k \pmod{p}$, which is a contradiction since $1 \leq k \leq p-1$. Consequently, the arithmetic means of the elements of S and T are not equal. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)