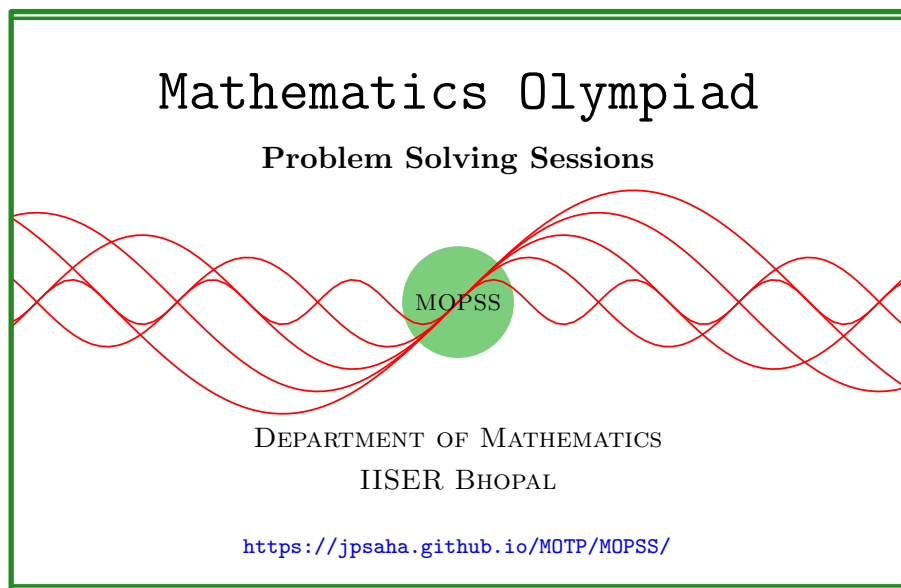


MOPSS

04 April 2026



Suggested readings

- Evan Chen's advice [On reading solutions](https://blog.evanchen.cc/2017/03/06/on-reading-solutions/), available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
- Evan Chen's [Advice for writing proofs/Remarks on English](https://web.evanchen.cc/handouts/english/english.pdf), available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- [Notes on proofs](#) by Evan Chen from [OTIS Excerpts](#) [[Che25](#), Chapter 1].
- [Tips for writing up solutions](https://www.math.utoronto.ca/barbeau/writingup.pdf) by Edward Barbeau, available at <https://www.math.utoronto.ca/barbeau/writingup.pdf>.
- Evan Chen discusses why [math olympiads](#) are a valuable experience for high schoolers in the post on [Lessons from math olympiads](#), available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

List of problems and examples

1.1	Exercise (Canadian Junior Mathematical Olympiad 2022 P2, AoPS)	2
1.2	Example	3
1.3	Exercise (Junior Balkan MO 2023 P3, AoPS, proposed by Théo Lenoir, France)	3
1.4	Exercise (Romania JBMO TST 2024 Day 2 P1, AoPS, proposed by Cristi Săvescu)	4
1.5	Exercise (Balkan Mathematical Olympiad 2018 P2, AoPS, proposed by Jeremy King, UK)	5

§1

Exercise 1.1 (Canadian Junior Mathematical Olympiad 2022 P2, AoPS). You have an infinite stack of T -shaped tetrominoes (composed of four squares of side length 1), and an $n \times n$ board. You are allowed to place some tetrominoes on the board, possibly rotated, as long as no two tetrominoes overlap and no tetrominoes extend off the board. For which values of n can you cover the entire board?

Walkthrough —

(a)

Solution 1. Note that a 4×4 board can be covered by T -shaped tetrominoes, as shown in Fig. 1. Hence, an $n \times n$ board can be covered by T -shaped tetrominoes if n is a multiple of 4.

The following claim proves the converse.

Claim — An $n \times n$ board can be covered by T -shaped tetrominoes only if n is a multiple of 4.

Proof of the claim. Assume that an $n \times n$ board can be covered by T -shaped tetrominoes. Each T -shaped tetromino covers exactly 4 unit squares, so the total number of unit squares covered must be a multiple of 4. Since the board has n^2 unit squares, it follows that n^2 must be a multiple of 4. This implies that n itself must be a multiple of 2.

Next, we will show that n is a multiple of 4. Consider the coloring of the board in a checkerboard pattern, where the top-left square is colored black. Each T -shaped tetromino covers either three black squares and one white square, or three white squares and one black square, depending on its orientation. Therefore, each T -shaped tetromino covers an odd number of squares of each color. Since n is even, the board has an even number of black squares and an

even number of white squares. To cover the entire board, we would need to cover an even number of black squares and an even number of white squares, and hence, we would need to use an even number of T -shaped tetrominoes. This implies that the total number of squares covered, which is 4 times the number of tetrominoes, must be a multiple of 8. Therefore, n^2 must be a multiple of 8, which implies that n must be a multiple of 4. \square

This proves that an $n \times n$ board can be covered by T -shaped tetrominoes if and only if n is a multiple of 4. \blacksquare

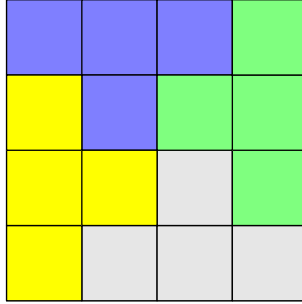


Figure 1: Canada Junior Mathematical Olympiad 2022 P2, Exercise 1.1

Example 1.2. Assume that on an 8×8 chessboard, two squares of different colors are removed. Prove that the remaining 62 squares can be covered by 31 dominoes, where each domino covers exactly two adjacent squares.

Exercise 1.3 (Junior Balkan MO 2023 P3, AoPS, proposed by Théo Lenoir, France). Alice and Bob play the following game on a 100×100 grid, taking turns, with Alice starting first. Initially the grid is empty. At their turn, they choose an integer from 1 to 100^2 that is not written yet in any of the cells and choose an empty cell, and place it in the chosen cell. When there is no empty cell left, Alice computes the sum of the numbers in each row, and her score is the maximum of these 100 numbers. Bob computes the sum of the numbers in each column, and his score is the maximum of these 100 numbers. Alice wins if her score is greater than Bob's score, Bob wins if his score is greater than Alice's score, otherwise no one wins.

Find if one of the players has a winning strategy, and if so which player has a winning strategy.

Walkthrough —

(a)

Solution 2. Let us cover each of the 100 rows using 50 non-overlapping 1×2 dominoes. The two dominoes covering the four leftmost cells of the first row

are called the *first* and *second* domino. We will show that Bob has a winning strategy, by using the following pairing strategy:

- If Alice places a number x in any cell of a domino other than the first and second domino, then Bob places $100^2 + 1 - x$ in the other cell of the same domino.
- If Alice places a number x in some cell of any of the first and second domino, then Bob places $100^2 + 1 - x$ in some cell of the other domino.

Note that Bob can always follow this strategy. Indeed, if Alice places a number x in a cell of a domino, then no cell of the grid filled earlier contains $100^2 + 1 - x$. Now, at the end of the game, the sum of the numbers in each row is $50 \cdot (100^2 + 1)$. On the other hand, the sum of the numbers in some column is at least $50 \cdot (100^2 + 1)$. Therefore, Bob's score is at least as large as Alice's score.

We will show that Bob's score is strictly greater than Alice's score. Let us assume for the sake of contradiction that Bob's score is equal to Alice's score, which is $50 \cdot (100^2 + 1)$. Then, the sum of the numbers in each column is $50 \cdot (100^2 + 1)$. In particular, the sum of the numbers in each of the first two columns is $50 \cdot (100^2 + 1)$. Hence, the sum of the numbers in the first domino is

$$2 \cdot 50 \cdot (100^2 + 1) - 99 \cdot (100^2 + 1) = 100^2 + 1.$$

Note that Bob's strategy implies that the sum of the numbers in any of the first and second domino is not $100^2 + 1$. This is a contradiction, which completes the proof of the fact that Bob has a winning strategy. ■

Remark. Dragomir Grozev argues that for the analogous problem on an $n \times n$ grid with n odd, Alice has a winning strategy.

Exercise 1.4 (Romania JBMO TST 2024 Day 2 P1, AoPS, proposed by Cristi Săvescu). The integers from 1 to 49 are written in a 7×7 table, such that for any $k \in \{1, 2, \dots, 7\}$ the product of the numbers in the k -th row equals the product of the numbers in the $(8 - k)$ -th row.

- Prove that there exists a row such that the sum of the numbers written on it is a prime number.
- Give an example of such a table.

Walkthrough —

(a)

Solution 3. An example of such a table is given in Fig. 2. ■

23	7 · 7	9 · 5	4 · 11	2 · 13	12 · 3	2
19	3 · 7	4 · 5	5	8 · 3	3 · 3	4
17	5 · 7	6 · 5	2 · 11	3	16 · 3	16
13	29	31	37	41	43	47
2 · 17	6 · 7	5 · 5	11	2 · 3	4 · 3	32
2 · 19	4 · 7	3 · 5	2 · 5	9 · 3	1	8
2 · 23	7	8 · 5	3 · 11	3 · 13	6 · 3	2 · 7

Figure 2: Romania JBMO TST 2024 Day 2 P1, Exercise 1.4

Exercise 1.5 (Balkan Mathematical Olympiad 2018 P2, AoPS, proposed by Jeremy King, UK). Let q be a positive rational number. Two ants are initially at the same point X in the plane. In the n -th minute ($n = 1, 2, \dots$) each of them chooses whether to walk due north, east, south or west and then walks the distance of q^n metres. After a whole number of minutes, they are at the same point in the plane (not necessarily X), but have not taken exactly the same route within that time. Determine all possible values of q .

Walkthrough —

(a)

Solution 4. Let (a, b) denote the coordinates of the point X . Let (x_n, y_n) and (x'_n, y'_n) be the coordinates of the two ants after n minutes. We have $x_0 = x'_0 = a$ and $y_0 = y'_0 = b$. Then for each n , note that $(x_n, y_n) - (x_{n-1}, y_{n-1})$ is equal to one of $(0, q^n), (q^n, 0), (0, -q^n), (-q^n, 0)$. Similarly, $(x'_n, y'_n) - (x'_{n-1}, y'_{n-1})$ is equal to one of $(0, q^n), (q^n, 0), (0, -q^n), (-q^n, 0)$. Let r be a positive integer such that $(x_r, y_r) = (x'_r, y'_r)$. Then we have

$$(a, b) + \sum_{n=1}^r (x_n - x_{n-1}, y_n - y_{n-1}) = (a, b) + \sum_{n=1}^r (x'_n - x'_{n-1}, y'_n - y'_{n-1}),$$

which implies that

$$\sum_{n=1}^r (x_n - x_{n-1}, y_n - y_{n-1}) = \sum_{n=1}^r (x'_n - x'_{n-1}, y'_n - y'_{n-1}).$$

This shows that

$$\sum_{n=1}^r ((x_n + iy_n) - (x_{n-1} + iy_{n-1})) = \sum_{n=1}^r ((x'_n + iy'_n) - (x'_{n-1} + iy'_{n-1})).$$

For $1 \leq n \leq r$, write

$$(x_n + iy_n) - (x_{n-1} + iy_{n-1}) = q^n i^{k_n}$$

and

$$(x'_n + iy'_n) - (x'_{n-1} + iy'_{n-1}) = q^n i^{k'_n},$$

where k_n and k'_n are integers. Also observe that k_n and k'_n are not congruent modulo 4 for some n with $1 \leq n \leq r$, since the two ants do not take the same route within r minutes. Then we have

$$\sum_{n=1}^r q^n i^{k_n} = \sum_{n=1}^r q^n i^{k'_n},$$

which implies that

$$\sum_{n=1}^r q^{n-1} (i^{k_n} - i^{k'_n}) = 0.$$

Note that $i^{k_n} - i^{k'_n}$ is divisible by $1 + i$, and the quotient is an element of $\{0, \pm 1, \pm i, \pm 1 \pm i\}$ for each n with $1 \leq n \leq r$. Hence, there exists a nonzero polynomial P of degree at most $r - 1$ with coefficients in $\{0, \pm 1, \pm i, \pm 1 \pm i\}$ such that $P(q) = 0$. Write $q = \frac{a}{b}$ with $a, b \in \mathbb{Z}_{\geq 1}$ and $\gcd(a, b) = 1$. Since q is nonzero, we may (and do) assume that P has a nonzero constant term. Also note that P is not a constant polynomial, since q is not zero. Considering the real and imaginary parts of the coefficients of P separately, and applying the rational root theorem to the resulting polynomials with integer coefficients, it follows that a divides the constant term of P in $\mathbb{Z}[i]$, and b divides the leading coefficient of P in $\mathbb{Z}[i]$. This implies that the square of the absolute value of a divides the square of the absolute value of the constant term of P , and the square of the absolute value of b divides the square of the absolute value of the leading coefficient of P . It follows that the square of the absolute value of each of a and b is at most 2, and hence $a = b = 1$. This shows that $q = 1$ is the only possible value of q .

Finally, we verify that $q = 1$ indeed works. Let the first ant walk north, east in the first two minutes, and the second ant walk east, north in the first two minutes. Then they are at the same point after two minutes, but have not taken the same route within that time. ■

References

- [Che25] EVAN CHEN. *The OTIS Excerpts*. Available at <https://web.evanchen.cc/excerpts.html>. 2025, pp. vi+289 (cited p. 1)