MOPSS

27 September 2025

Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [Che25, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1

Exercise 1.1 (SMMC 2017 A1, AoPS)**.** The five sides and five diagonals of a regular pentagon are drawn on a piece of paper. Two people play a game, in which they take turns to colour one of these ten line segments. The first player colours line segments blue, while the second player colours line segments red. A player cannot colour a line segment that has already been coloured. A player wins if they are the first to create a triangle in their own colour, whose three vertices are also vertices of the regular pentagon. The game is declared a draw if all ten line segments have been coloured without a player winning. Determine whether the first player, the second player, or neither player can force a win.

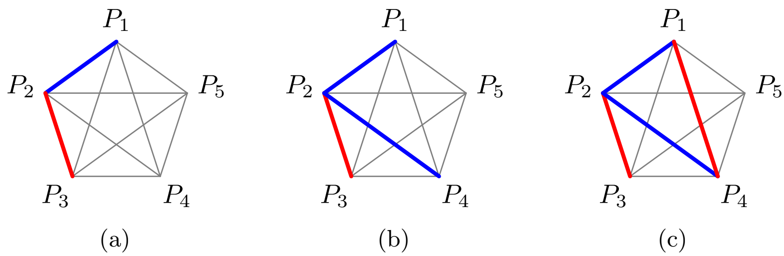


Figure 1: SMMC 2017 A1, Exercise 1.1, First two colored segments share a common vertex

Walkthrough — To force a win, the first player colors a side of the pentagon in blue in the first move.

Let us first consider the case that the two edges coloured in the first two

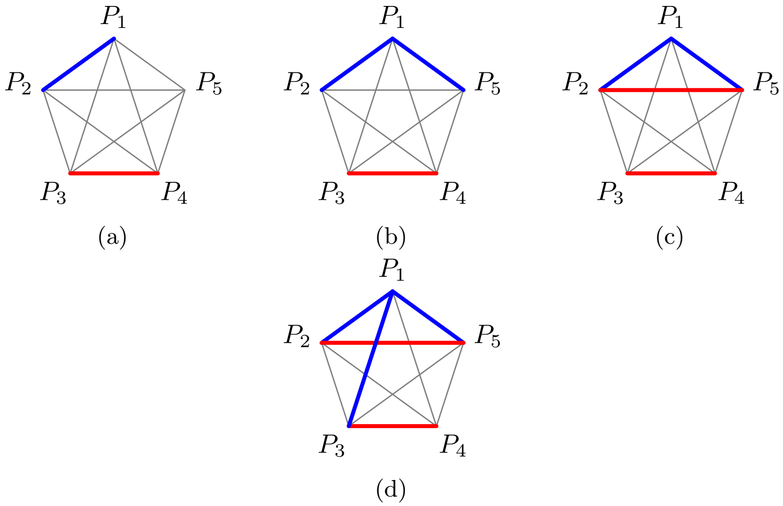


Figure 2: SMMC 2017 A1, Exercise 1.1, First two colored segments share no common vertex

moves have a common endpoint.

(a) Show that there is no loss of generality in assuming that the first player colours P_1P_2 in blue and the second player colours P_2P_3 in red, where $P_1P_2P_3P_4P_5$ denotes the pentagon.

(b) Next, the first player colours P_2P_4 in blue.

(c) If the second player does not color P_1P_4, then in the next move the first player colours P_1P_4 in blue and wins. It remains to consider the case that the second player colours P_1P_4 in red, which we assume from now on.

(d) The first player colours P_2P_5 in blue.

(e) During the next turn of the first player, one of the edges P_1P_5 and P_4P_5 is coloured in blue, thus creating a blue triangle.

We are still left with the case that the two edges coloured in the first two moves do not have a common endpoint.

(a) Show that there is no loss of generality in assuming that the first player colours P_1P_2 in blue and the second player colours P_3P_4 in red.

(b) Next, the first player colours P_1P_5 in blue.

(c) If the second player does not color P_2P_5, then in the next move the first player colours P_2P_5 in blue and wins. It remains to consider the case that the second player colours P_2P_5 in red, which we assume from now on.

(d) The first player colours P_1P_3 in blue.

(e) During the next turn of the first player, one of the edges P_2P_3 and P_3P_5 is coloured in blue, thus creating a blue triangle.

Solution 1. ■

Exercise 1.2 (Tournament of Towns Fall 2013, Senior A Level P4, AoPS)**.** Integers $1, 2, \ldots, 100$ are written on a circle, not necessarily in that order. Can it be that the absolute value of the difference between any two adjacent integers is at least 30 and at most 50?

Walkthrough —

(a) Assume that such an arrangement is possible.

(b) Consider the integers $1, 2, \ldots, 25, 76, 77, \ldots, 100$.

(c) Show that no two of these integers are adjacent.

(d) Conclude that the integers $1, 2, \ldots, 25, 76, 77, \ldots, 100$ and $26, 27, \ldots, 75$ are placed alternately along the circle.

(e) Show that 26 is adjacent to 76 only.

(f) Derive a contradiction.

Solution 2. Let us assume that the integers $1, 2, \ldots, 100$ can be arranged along the circumference of a circle in some order such that the absolute value of the difference between any two adjacent integers is at least 30 and at most 50.

Since the difference of any two of the integers $1, 2, \ldots, 25, 76, 77, \ldots, 100$ is less than 30 or greater than 50 it follows that no two of these integers are adjacent. Consequently, the elements of the sets

$$\{1, 2, \ldots, 25, 76, 77, \ldots, 100\}, \{26, 27, \ldots, 75\}$$

are placed alternately along the circle. However, 26 is adjacent to 26 only, which is impossible. This shows that there is no arrangement of the integers $1, 2, \ldots, 100$ along the circumference of a circle satisfying the given condition. ■

Example 1.3. Can a 12×12 chessboard be covered by non-overlapping trominos? A tromino is an L-shaped piece consisting of three squares, where these squares are of the same size as the smallest squares in the chessboard.

Walkthrough —

(a) Does Fig. 3 help?

Solution 3. Note that a 2×3 rectangle can be tiled by two non-overlapping trominos. Since a 12×12 chessboard can be divided into six 2×3 rectangles, it follows that a 12×12 chessboard can be tiled by non-overlapping trominos. ■

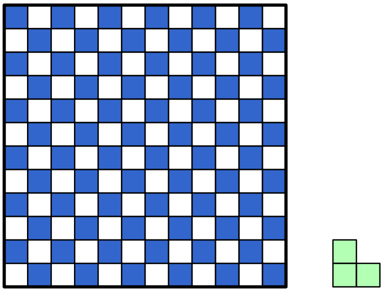
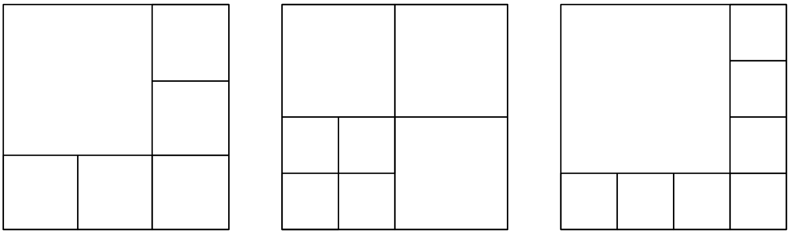


Figure 3: Tiling a 12×12 chessboard using trominos.



(a) Tiling using 6 squares. (b) Tiling using 7 squares. (c) Tiling using 8 squares.

Figure 4: Tiling a square using 6, 7, and 8 squares.

Example 1.4. For any integer $n \geq 6$, show that a square can be cut into n squares, not necessarily of the same size.

Walkthrough —

(a) Try to work it out for small values of n, for instance $n = 6, 7, 8, 9$.

(b) What happens if you want to go from n to $n + 3$?

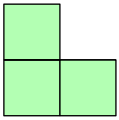
Solution 4. Note that if we can do it for n, then we can do it for $n + 3$ by cutting one of the squares into four smaller squares. So it suffices to do it for $n = 6, 7, 8$, which is shown in Fig. 4. By induction, we are done. ■

Example 1.5. Show that for any integer $n \geq 1$, a $2^n \times 2^n$ chessboard with one square removed can be tiled by non-overlapping trominos, that is, L-shaped pieces consisting of three squares.

Walkthrough —

(a) Can a 2×2 chessboard with one square removed be tiled by trominos?

(b) Can a 4×4 chessboard with one square removed be tiled by trominos?



(a)

Figure 5: Tiling a $2^n \times 2^n$ chessboard with one square removed using trominos.

(c) Can a 8×8 chessboard with one square removed be tiled by trominos?

(d) What about tiling a $2^n \times 2^n$ chessboard with one square removed using trominos for $n \geq 4$? Does induction help?

Solution 5. ■

Exercise 1.6 (Tournament of Towns Spring 2025, Senior O Level P1, AoPS, by Mark Alexeev)**.** Find the minimum positive integer such that some four of its natural divisors sum up to 2025.

Walkthrough —

(a) If n is a positive integer and a, b, c, and d are four distinct positive divisors of n satisfying $a < b < c < d$, then observe that the inequalities

$$d \leq n, c \leq \frac{n}{2}, b \leq \frac{n}{3}, a \leq \frac{n}{4}$$

hold.

Solution 6. Suppose there exists a positive integer n such that some four of its natural divisors sum up to 2025. Let these four divisors be a, b, c, and d. Without loss of generality[1], we can assume that $a < b < c < d$. Note that

$$d \leq n, c \leq \frac{n}{2}, b \leq \frac{n}{3}, a \leq \frac{n}{4},$$

which shows that

$$2025 = a + b + c + d \leq n + \frac{n}{2} + \frac{n}{3} + \frac{n}{4} = \frac{25n}{12}.$$

This implies that

$$n \geq \frac{12 \cdot 2025}{25} = 972.$$

Note that the positive integers $972, 486, 324, 243$ divide 972 and sum up to 2025.

This proves that 972 is the smallest positive integer such that some four of its natural divisors sum up to 2025. ■

[1] Why there is no loss of generality in assuming $a < b < c < d$?

Exercise 1.7 (SMMC 2017 B2, AoPS)**.** Determine all pairs (p, q) of positive integers such that p and q are primes, and $p^{q-1} + q^{p-1}$ is the square of an integer.

Walkthrough —

(a) Check that $(p, q) = (2, 2)$ works.

(b) If p, q are odd, show that $p^{q-1} + q^{p-1} \equiv 2 \pmod 4$, so it cannot be a perfect square.

(c) Without loss of generality, let p be an odd prime, and $q = 2$. Then $p^{q-1} + q^{p-1} = 2^{p-1} + p$.

(d) Write $2^{p-1} + p = m^2$.

(e) Factorize to obtain $(m - 2^{\frac{p-1}{2}})(m + 2^{\frac{p-1}{2}}) = p$, which gives

$$m + 2^{\frac{p-1}{2}} = p.$$

(f) Note that $p < 2^{\frac{p-1}{2}}$ holds[a] for $p \geq 7$.

(g) Conclude that p can only be 3 or 5.

[a]Try to show that $2^{\frac{n-1}{2}} > n$ holds for all integers $n \geq 7$.

Solution 7. First, note that $(p, q) = (2, 2)$ works since $2^{2-1} + 2^{2-1} = 4 = 2^2$.

If p and q are odd primes, then both $p - 1$ and $q - 1$ are even, and hence,

$$p^{q-1} + q^{p-1} \equiv 1 + 1 \equiv 2 \pmod 4$$

holds. Since no perfect square is congruent to 2 modulo 4, it follows that not both of p and q are odd.

Without loss of generality[2], let p be an odd prime, and $q = 2$. Then

$$p^{q-1} + q^{p-1} = p^{2-1} + 2^{p-1} = p + 2^{p-1}.$$

Write

$$p + 2^{p-1} = m^2$$

for some positive integer m. Rearranging gives

$$m^2 - 2^{p-1} = p,$$

which yields the factorization

$$(m - 2^{\frac{p-1}{2}})(m + 2^{\frac{p-1}{2}}) = p.$$

Since p is prime and

$$m - 2^{\frac{p-1}{2}} < m + 2^{\frac{p-1}{2}},$$

[2]Why there is no loss of generality?

we obtain that
$$m + 2^{\frac{p-1}{2}} = p,$$
Using the following claim, we conclude that p can only be 3 or 5. If $p = 3$, then $p^{q-1} + q^{p-1} = 3^{2-1} + 2^{3-1} = 3 + 4 = 7$. If $p = 5$, then $p^{q-1} + q^{p-1} = 5^{2-1} + 2^{5-1} = 5 + 16 = 21$. Thus, the only solution is $(p, q) = (2, 2)$.

Claim — For any positive integer $n \geq 7$, the inequality
$$2^{\frac{n-1}{2}} > n$$
holds.

Proof. We prove this by induction. For $n = 7$, we have $2^{\frac{7-1}{2}} = 2^3 = 8 > 7$. Now, assume that the inequality holds for some $k \geq 7$. Note that
$$2^{\frac{(k+1)-1}{2}} = 2^{\frac{k-1}{2}+\frac{1}{2}} = \sqrt{2} \cdot 2^{\frac{k-1}{2}} > \sqrt{2} \cdot k$$
holds. Observe that
$$\begin{aligned}(\sqrt{2}k)^2 - (k+1)^2 &= 2k^2 - (k^2 + 2k + 1) \\ &= k^2 - 2k - 1 \\ &= (k-1)(k-3) - 2 \\ &\geq (7-1)(7-3) - 2 \\ &> 0.\end{aligned}$$
This shows that
$$2^{\frac{(k+1)-1}{2}} > \sqrt{2} \cdot k > k + 1.$$
Thus, by induction, the claim holds for all integers $n \geq 7$. □

■

Exercise 1.8 (SMMC 2020 A1, AoPS)**.** There are 1001 points in the plane such that no three are collinear. The points are joined by 1001 line segments such that each point is an endpoint of exactly two of the line segments. Prove that there does not exist a straight line in the plane that intersects each of the 1001 line segments in an interior point. An **interior point** of a line segment is a point of the line segment that is not one of the two endpoints.

Walkthrough —

(a) On the contrary, let us assume that there exists a straight line L in the plane that intersects each of the 1001 line segments in an interior point.

(b) Show that L cannot pass through any of the 1001 points.

(c) Color the points on one side of L red, and the points on the other side of L blue.

> **(d)** Show that the number of the line segments is equal to twice the number of red points, to obtain a contradiction.

Solution 8. On the contrary, let us assume that there exists a straight line L in the plane that intersects each of the 1001 line segments in an interior point. Note that L cannot pass through any of the 1001 points. Indeed, if L passes through one of the 1001 points, then the two line segments that have this point as an endpoint, cannot intersect L at their interior points, since no three of the given points are collinear. Let us color the points on one side of L red, and the points on the other side of L blue. Since L intersects each of the 1001 line segments in an interior point, each of the 1001 line segments has one red endpoint and one blue endpoint. Moreover, each of the 1001 points is an endpoint of exactly two of the line segments. Therefore, the number of line segments is equal to twice the number of red points, which is impossssible since 1001 is odd. This completes the proof. ■

Example 1.9. Let $a_1, a_2, \dots, a_n$ be positive integers. Let b_k denote the number of those a_i for which a_i is greater than or equal to k. Prove that

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_{a_n}.$$

> **Walkthrough** —
>
> **(a)** Try to interpret both sides of the equation in terms of counting objects.
>
> **(b)** Can you think of a way to represent the integers $a_1, a_2, \dots, a_n$ using dots?

Solution 9. Consider the set

$$S = \{(i,j) : 1 \leq i \leq n, 1 \leq j \leq a_i\}.$$

The size of this set is equal to $a_1 + a_2 + \cdots + a_n$. On the other hand, for any positive integer j, the number of i such that $(i,j) \in S$ is exactly b_j. This shows that the size of S is also equal to $b_1 + b_2 + \cdots + b_{a_n}$. This yields the desired result.

Alternatively, note that

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &= \sum_{i=1}^{n} \sum_{j=1}^{a_i} 1 \\ &= \sum_{i=1}^{n} \sum_{j=1}^{a_n} 1_{a_i \geq j}(i,j) \\ &= \sum_{j=1}^{a_n} \sum_{i=1}^{n} 1_{a_i \geq j}(i,j) \end{aligned}$$

$$= \sum_{j=1}^{a_n} b_j,$$

where $1_{a_i \geq j}(i, j)$ is 1 if $a_i \geq j$, and 0 otherwise. ■

> **Remark.** Note that the above two solutions are essentially the same. The first solution uses a combinatorial argument, while the second solution uses a more algebraic approach. The key idea in both solutions is to count the same set S in two different ways.

Exercise 1.10 (BStat-BMath 2014, AoPS)**.** A class has 100 students. Let a_i, $1 \leq i \leq 100$, denote the number of friends the i-th student has in the class. For each $0 \leq j \leq 99$, let c_j denote the number of students having at least j friends. Show that

$$a_1 + a_2 + \cdots + a_{100} = c_1 + c_2 + \cdots + c_{99}.$$

> **Walkthrough** —
>
> **(a)** Does solving the previous problem help?

Solution 10. For $1 \leq i \leq 100$, denote the i-th student by s_i. For $1 \leq j \leq 99$, let C_j denote the set of students having at least j friends. Note that for any $1 \leq i \leq 100$,

$$a_i = \sum_{j=1}^{99} 1_{C_j}(s_i)$$

holds, where for $1 \leq j \leq 99$, 1_{C_j} denotes the map, defined on $\{s_1, s_2, \ldots, s_{100}\}$, given by

$$1_{C_j}(s_i) = \begin{cases} 1 & \text{if } s_i \text{ lies in } C_j, \\ 0 & \text{otherwise.} \end{cases}$$

Summing over $1 \leq i \leq 100$, and interchanging the order of summation, we obtain

$$\begin{aligned} a_1 + a_2 + \cdots + a_{100} &= \sum_{j=1}^{99} \sum_{i=1}^{100} 1_{C_j}(s_i) \\ &= \sum_{j=1}^{99} |\{s_i \mid s_i \in C_j\}| \\ &= \sum_{j=1}^{99} c_j. \end{aligned}$$

This completes the proof. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)