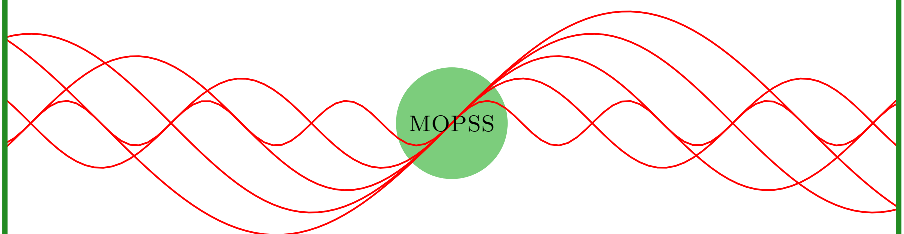
MOPSS

11 October 2025

Mathematics Olympiad

Problem Solving Sessions



Department of Mathematics

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1

Exercise 1.1 (British Mathematical Olympiad Round 2 2025 P1)**.** Prove that if n is a positive integer, then $\frac{1}{n}$ can be expressed as a finite sum of reciprocals of different triangular numbers. (A **triangular number** is an integer which is equal to $\frac{k(k+1)}{2}$ for some positive integer k.)

Walkthrough —

(a) Note that it suffices to express $\frac{1}{2n}$ as a finite sum of reciprocal of consecutive integers.

(b) Observe that

$$\frac{1}{2n} = \frac{1}{n} - \frac{1}{2n}$$

holds.

Solution 1. Note that

$$\frac{1}{2n} = \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \left(\frac{1}{2n+1} - \frac{1}{2n}\right)$$

holds. This yields that $\frac{1}{n}$ is the sum of the reciprocals of

$$\frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2}, \ldots, \frac{(2n+1)2n}{2},$$

which are distinct triangular numbers since the map $x \mapsto \frac{x(x+1)}{2}$ from $\mathbb{R} \to \mathbb{R}$ is injective. ■

Exercise 1.2 (British Mathematical Olympiad Round 1 2019 P1)**.** Show that there are at least three prime numbers p less than 200 for which $p+2$, $p+6$, $p+8$ and $p+12$ are all prime. Show also that there is only one prime number q for which $q+2$, $q+6$, $q+8$, $q+12$ and $q+14$ are all prime.

Walkthrough —

(a) Let p be a prime number such that p, $p+2$, $p+6$, $p+8$ and $p+12$ are all prime.

(b) Show that p is odd.

(c) Using that $p+6, p+8$ are primes greater than 3, show that $p \equiv 2 \pmod 3$.

(d) Using that $p+6, p+8, p+12$ are primes greater than 5, show that $p = 5$ or $p \equiv 1 \pmod 5$.

(e) Using that $p+2, p+6, p+8, p+12$ are primes greater than 7, show that $p = 7$ or p is congruent to either 3 or 4 modulo 7.

(f) Using the above congruence conditions modulo 3, 5 and 7, show that if $p \leq 200$, then p is equal to one of the integers

$$11, 41, 71, 101, 131, 161, 191.$$

(g) Complete the first part of the problem.

(h) For the second part of the problem, show that if q is a prime number such that q, $q+2$, $q+6$, $q+8$, $q+12$ and $q+14$ are all prime, then q is at most 5.

Solution 2. Let p be a prime number such that p, $p+2$, $p+6$, $p+8$ and $p+12$ are all prime. Note that p is odd. Since $p+6, p+8$ are primes greater than 3, it follows that

$$p \not\equiv 0 \pmod 3, \quad p \not\equiv 1 \pmod 3,$$

which implies that $p \equiv 2 \pmod 3$. Since $p+6, p+8, p+12$ are primes greater than 5, it follows that

$$p \not\equiv 4 \pmod 5, \quad p \not\equiv 2 \pmod 5, \quad p \not\equiv 3 \pmod 5,$$

which implies that $p = 5$ or $p \equiv 1 \pmod 5$. Note that if $p = 5$, then $p+2, p+6, p+8, p+12$ are all prime. Henceforth, let us assume that $p \equiv 1 \pmod 5$. Note that $p+2, p+6, p+8, p+12$ are all primes greater than 7. It follows that p not congruent to any of $5, 1, 6, 2$ modulo 7. This shows that $p = 7$ or p is congruent to either 3 or 4 modulo 7. Note that if $p = 7$, then $p+2$ is not a prime. This implies that p is congruent to either 3 or 4 modulo 7. Since $p \equiv 2 \pmod 3$ and $p \equiv 1 \pmod 5$, it follows that p is congruent to 11 modulo 15. If $p \leq 200$, then p is equal to one of the integers

$$11, 41, 71, 101, 131, 161, 191.$$

Among the above integers, only $11, 101$ satisfy the required congruence condition modulo 7. Note that if $p = 11$, then $p+2, p+6, p+8, p+12$ are all prime. Moreover, if $p = 101$, then $p+2, p+6, p+8, p+12$ are all prime. This shows that there are precisely three prime numbers p less than 200 for which $p+2, p+6, p+8, p+12$ are all prime, and these are $5, 11, 101$.

Let q be a prime number such that $q, q+2, q+6, q+8, q+12, q+14$ are all prime. Since $q+6, q+8$ are primes greater than 3, it follows that $q \equiv 2 \pmod 3$. Note that q is odd, and hence, q is at least 5. Since $q+6, q+8, q+12, q+14$ are primes greater than 5, it follows that q is equal to 5. Note that if $q = 5$, then $q+2, q+6, q+8, q+12, q+14$ are all prime. This shows that there is only one prime number q for which $q+2, q+6, q+8, q+12, q+14$ are all prime, and this is 5. ■

Exercise 1.3 (British Mathematical Olympiad Round 1 2016/17 P3). Determine all pairs (m, n) of positive integers which satisfy the equation $n^2 - 6n = m^2 + m - 10$.

Walkthrough —

(a) Complete the square on both sides, and rearrange to get a suitable factorization.

Solution 3. Let m, n be positive integers satisfying

$$n^2 - 6n = m^2 + m - 10.$$

This yields

$$(n-3)^2 = m^2 + m - 1,$$

which implies

$$(n-3)^2 = \left(m + \frac{1}{2}\right)^2 - \frac{5}{4}.$$

Rearranging gives

$$(2m+1)^2 - (2(n-3))^2 = 5.$$

Factorizing yields

$$(2m+1-2(n-3))(2m+1+2(n-3)) = 5.$$

Using $n^2 - 6n = m^2 + m - 10$, it follows that n is not an odd integer. Note that if $n = 2$, then the given equation holds yields $m = 1$. It remains to consider the case that $n \geq 4$, which we assume from now on. Note that

$$2m+1-2(n-3) \leq 2m+1+2(n-3)$$

holds, and hence we obtain

$$2m+1-2(n-3) = 1, \quad 2m+1+2(n-3) = 5.$$

This gives $m = 1, n = 4$.

Moreover, if $m = 1, n = 4$, then $n^2 - 6n = m^2 + m - 10$ holds. Further, if $m = 1, n = 2$, then also the given equation holds.

Therefore, the pairs (m, n) of positive integers which satisfy the equation $n^2 - 6n = m^2 + m - 10$ are precisely $(1, 2)$ and $(1, 4)$. ■

Exercise 1.4 (British Mathematical Olympiad Round 1 2023 P4). Find all positive integers n such that $n \times 2^n + 1$ is a perfect square.

Walkthrough —

(a) Write $n \times 2^n + 1 = k^2$ for some positive integer k.

(b) Rearranging gives $n \times 2^n = k^2 - 1 = (k-1)(k+1)$.

(c) Show that $k-1$ and $k+1$ are consecutive even integers.

(d) Using that one of $k-1$ and $k+1$ is not divisible by 4, show that one of $k-1$ and $k+1$ is divisible by 2^{n-1}.

(e) Deduce that $n \leq 4$.

Solution 4. Let n be a positive integer such that $n \times 2^n + 1 = k^2$ for some positive integer k. Rearranging gives $n \times 2^n = k^2 - 1 = (k-1)(k+1)$. Since the product $(k-1)(k+1)$ of the consecutive integers $k-1$ and $k+1$ is even, it follows that $k-1, k+1$ are both even. Thus, we can write $k-1 = 2a$ and $k+1 = 2b$ for some positive integers a and b with $b = a+1$. Note that the greatest common divisor of the integers $k-1$ and $k+1$ is 2. Therefore, the greatest common divisor of the integers a and b is 1. This shows that at least one of the integers a and b is odd. Consequently, one of the integers $k-1$ and $k+1$ is divisible by 2^{n-1}. This shows that

$$n \times 2^n \geq 2^{n-1}(2^{n-1} - 2)$$

holds, which simplifies to $n \geq 2^{n-2} - 1$. Using induction, it follows that $2^{m-2} > m + 1$ holds for all integers $m \geq 5$. Therefore, we have $n \leq 4$. Note that n is not equal to any of the integers $1, 4$. Moreover, if $n = 2$, then $n \times 2^n + 1 = 9$, which is a perfect square. Finally, if $n = 3$, then $n \times 2^n + 1 = 25$, which is a perfect square. Thus, the only positive integer n such that $n \times 2^n + 1$ is a perfect square is $n = 2$ and $n = 3$. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)