MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [Che25, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Part A

Example 1.1 (Bay Area MO 2000 P1)**.** Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1.

> **Walkthrough** — Consider the case of an odd integer, the case of a multiple of 4, and the case of an even integer, which is not a multiple of 4.

Solution 1. Note that any odd integer can be expressed as the sum of two relatively prime integers. Indeed, for any integer n, the integer $2n + 1$ is the sum of the relatively prime integers $n, n + 1$.

For any integer k, note that

$$4k = (2k - 1) + (2k + 1)$$

holds, and the integers $2k - 1, 2k + 1$ are relatively prime since any of their common divisors is odd and divides $(2k + 1) - (2k - 1) = 2$.

For any integer ℓ, note that

$$4\ell + 2 = (2\ell - 1) + (2\ell + 3)$$

holds, and the integers $2\ell - 1, 2\ell + 3$ are relatively prime since any of their common divisors is odd and divides $(2\ell + 3) - (2\ell - 1) = 4$. ■

Example 1.2 (Moscow MO 1973 Day 1 Grade 8 P4)**.** Prove that the equation

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{p},$$

where x, y are positive integers, has exactly 3 solutions if p is a prime and the number of solutions is greater than three if $p>1$ is not a prime. We consider solutions (a, b) and (b, a) for $a \neq b$ distinct.

> **Walkthrough** — Is the given equation equivalent to
>
> $$(x-p)(y-p)=p^{2}?$$

Example 1.3 (cf. Moscow MO 1973 Day 1 Grade 8 P4 Example 1.2)**.** For any positive integer n, show that the number of ordered pairs (x, y) of positive integers for which

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$$

is equal to the number of positive divisors of n^2.

> **Walkthrough** — Is the given equation equivalent to
>
> $$(x-n)(y-n)=n^{2}?$$

Solution 2. For positive integers x, y, note that

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$$

holds if and only if

$$(x-n)(y-n)=n^{2}$$

holds. Observe that if x, y are positive integers satisfying the given equation, then $x>n$ and $y>n$ holds. This shows that the solutions of the given equation over the positive integers are in one-to-one correspondence with pairs of positive integers (a, b) such that $ab=n^2$, through the map

$$(a+n, b+n) \leftrightarrow (a, b).$$

This completes the proof. ∎

Example 1.4 (India INMO 1991 P10, cf. Moscow MO 1973 Day 1 Grade 8 P4 Example 1.2)**.** For any positive integer n, let $S(n)$ denote the number of ordered pairs (x, y) of positive integers for which

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$$

(for instance, $S(2)=3$). Determine the set of positive integers n for which $S(n)=5$.

Walkthrough — Is the given equation equivalent to

$$(x-n)(y-n) = n^2?$$

Solution 3. For positive integers x, y, note that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

holds if and only if

$$(x-n)(y-n) = n^2$$

holds. Observe that if x, y are positive integers satisfying the given equation, then $x > n$ and $y > n$ holds. This shows that the solutions of the given equation over the positive integers are in one-to-one correspondence with the pairs of positive integers (a, b) such that $ab = n^2$, through the map

$$(a+n, b+n) \leftrightarrow (a, b).$$

Hence, the set of positive integers n satisfying $S(n) = 5$ is equal to the set of positive integers n such that n^2 has precisely 5 positive divisors. Note that any such integer n is larger than 1. Writing n as a product of powers of distinct primes, it follows that n^2 has precisely 5 positive divisors if and only if n is the square of a prime. Indeed, if $p_1, \dots, p_r$ are distinct primes, and $a_1, \dots, a_r$ are positive integers, then the integer $(p_1^{a_1} \dots p_r^{a_r})^2$ has precisely 5 positive divisors if and only if

$$(2a_1+1)(2a_2+1)\dots(2a_r+1) = 5$$

holds, which is equivalent to $r = 1, a_1 = 2$. This proves that the positive integers satisfying $S(n) = 5$ are precisely the squares of the primes. ■

Example 1.5 (India RMO 1992 P2, cf. Moscow MO 1973 Day 1 Grade 8 P4 Example 1.2)**.** If $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, where a, b, c are positive integers with no common factor, prove that $(a+b)$ is the square of an integer.

Walkthrough — Is the given equation equivalent to

$$(a-c)(b-c) = c^2?$$

Solution 4. Let a, b, c be positive integers satisfying the given equation. Assume that a, b have no common prime factors. Note that $(a-c)(b-c) = c^2$ holds. Also note that any common prime divisor of $a-c, b-c$ divides $(a-c)(b-c) = c^2$, and hence it divides the integers a, b, which is impossible. This shows that the integers $a-c, b-c$ are relatively prime, and satisfy $(a-c)(b-c) = c^2$. Note also that $a > c$ holds. Hence, there exist relatively prime positive integers x, y such that $c = xy$, $a - c = x^2$ and $b - c = y^2$ holds. This gives

$$a = c + x^2 = xy + x^2, \quad b = c + y^2 = xy + y^2.$$

This implies that $a + b$ is a perfect square. ■

Example 1.6 (UK BMO 2005 Round 2 P1, cf. Moscow MO 1973 Day 1 Grade 8 P4 Example 1.2)**.** The integer N is positive. There are exactly 2005 ordered pairs (x, y) of positive integers satisfying

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}.$$

Show that N is a perfect square.

Walkthrough —

(a) Is the given equation equivalent to

$$(x - N)(y - N) = N^2?$$

(b) Note that it suffices to show that if N^2 has precisely 2005 positive divisors for some positive integer N, then N is a perfect square.

(c) Note that $2005 = 5 \cdot 401$.

(d) Observe that 401 is a prime, and hence, all the prime factors of 2005 are congruent to 1 modulo 4.

Example 1.7 (India RMO 1994 P5)**.** Let A be a set of 16 positive integers with the property that the product of any two distinct numbers of A will not exceed 1994. Show that there are two numbers a and b in A which are not relatively prime.

Walkthrough —

(a) Suppose $\mathcal{A}$ is a set of 16 positive integers, and assume that $\mathcal{A}$ does not satisfy the conclusion of the problem, that is, it is false that **some two** *numbers in* $\mathcal{A}$ **are not** *relatively prime.* In other words, $\mathcal{A}$ has the property that **any two** *numbers in* $\mathcal{A}$ **are** *relatively prime.*

(b) Prove that such a set $\mathcal{A}$ would fail to satisfy the given condition, which states that *the product of* **any two** *distinct elements of* $\mathcal{A}$ *is smaller than* 1994. In other words, show that *the product of* **some two** *elements of* $\mathcal{A}$ *is greater than or equal to* 1994.

Remark. Observe that the above argument shows that if a set of 16 positive integers violates the conclusion of the problem, then it does not satisfy the given condition. **Convince yourself** that **doing so does prove** that if a set of 16 positive integers satisfy the given condition, then **there is no way that** it would fail to satisfy the conclusion.

Further, also **convince yourself** that establishing that the failure of the conclusion forces the failure of the given condition [a] is **equivalent to** establishing that the given condition implies the stated conclusion.

[a]When there are more than one condition, the given conditions are to be **considered together**, and the *failure of the* **totality** *of the given conditions* means the *failure of* **at least one of** *the given conditions.*

Solution 5. Note that if $n \geq 1$, and $a_1, \dots, a_n$ are distinct and pairwise coprime positive integers such that $a_i \geq 2$ for all i, then one of them admits a prime factor which is at least as large as the n-th prime. Indeed, for each i, if we fix a prime divisor p_i of a_i, then using that $a_1, \dots, a_n$ are pairwise coprime, it follows that $p_1, \dots, p_n$ are distinct primes, and hence the largest among them is at least as large as the n-th prime. Consequently, if $n \geq 2$, and $a_1, \dots, a_n$ are distinct and pairwise coprime positive integers, then at least $(n-1)$ of them are greater than 1, and hence one of them is divisible by a prime at least as large as the $(n-1)$-st prime.

If possible, let us assume that the elements of A are pairwise coprime. Hence, A contains an element x which is divisible by a prime at least as large as the 15th prime. Similarly, $A \setminus \{x\}$ has an element y which is divisible by a prime at least as large as the 14th prime. Since the 14th and 15th primes are $43, 47$ respectively, it follows that $x \geq 47, y \geq 43$, and consequently,

$$xy \geq 47 \cdot 43 = 2021 > 1994,$$

which contradicts the hypothesis. This proves that there are two numbers a and b in A which are not relatively prime. ■

Remark. Note that the above problem is similar to the following problem in spirit.

Example 1.8. For a set A of consisting of positive integers, let $\ell(A)$ denote the largest integer which can be expressed as the product of two distinct elements of A. What is the smallest element of the set which consists of the integers of the form $\ell(A)$ as A runs over the sets of size 16 and consisting of positive integers?

Bogus Solution. Since $\ell(A)$ is to be minimized as A runs over the sets 16 pairwise coprime integers and $\ell(A)$ denotes the maximum of the products of the pairs of elements of A, it follows that the minimum value of $\ell(A)$ is achieved precisely when the elements of A are as small as possible. This shows that the minimum value of $\ell(A)$ occurs when

$$A = \{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}.$$

Hence, the minimum value of $\ell(A)$ is $43 * 47 = 2021$.

Exercise 1.9. What goes wrong with the above?

Example 1.10 (India RMO 1994 P3)**.** Find all 6-digit natural numbers $a_1a_2a_3a_4a_5a_6$ formed by using the digits $1, 2, 3, 4, 5, 6$, once each such that the number $a_1a_2 \ldots a_k$ is divisible by k, for $1 \le k \le 6$.

Walkthrough —

(a) Show that a_2, a_4, a_6 are equal to $2, 4, 6$ in some order.

(b) Prove that $a_5 = 5$, and a_1, a_3 are equal to $1, 3$ in some order.

(c) Using that 3 divides $a_1a_2a_3$, determine a_2.

(d) Using the divisibility condition by 4, show that $a_4 = 6$, and conclude that $a_6 = 4$.

Solution 6. Since $a_1a_2, a_1a_2a_3a_4, a_1a_2a_3a_4a_5a_6$ are divisible by 2, it follows that a_2, a_4, a_6 are even, and hence they are equal to $2, 4, 6$ in some order. Using that $a_1a_2a_3a_4a_5$ is divisible by 5, we get that $a_5 = 5$. So a_1, a_3 are equal to $1, 3$ in some order. Using that $a_1a_2a_3$ is a multiple of 3, we obtain

$$a_1 + a_2 + a_3 \equiv 0 \bmod 3,$$

which yields

$$a_2 \equiv 2 \bmod 3,$$

and hence, $a_2 = 2$ holds. Note that $1234, 3214$ are not divisible by 4. This shows that $a_4 = 6$, and hence $a_6 = 4$. It follows that $a_1a_2a_3a_4a_5a_6$ is equal to 321654, or 123654. Note that the integers $321654, 123654$ satisfy the required conditions too. This proves that $321654, 123654$ are precisely all the 6-digit numbers satisfying the given condition. ■

Example 1.11 (Tournament of Towns, India RMO 1995 P3)**.** [**Tao06**, Problem 2.1] Prove that among any 18 consecutive three digit numbers there is at least one number which is divisible by the sum of its digits.

Walkthrough —

(a) Show that one among any such consecutive integers is divisible by 18.

(b) Prove that its sum of digits, is a multiple of 9, and conclude that it is equal to one of $9, 18, 27$.

(c) Show that the sum of its digits is not 27.

Solution 7. Note that among 18 consecutive three digit numbers, there is an integer divisible by 18. Denote it by $n = 100a + 10b + c$ with a, b, c denoting integers lying between 0 and 9. It follows that 9 divides n, and hence 9 divides $a + b + c$. This shows that $a + b + c$ is equal to one of $9, 18, 27$. Note that $a + b + c = 27$ holds only if $n = 999$. Since 18 divides n, it follows that $a + b + c \neq 27$, and hence, $a + b + c$ is equal to one of $9, 18$. This proves that $a + b + c$ divides n. ■

Example 1.12 (Australian MO 1982, India RMO 2004 P6)**.** Let $(p_1, p_2, p_3, \dots, p_n, \dots)$ be a sequence of primes, defined by $p_1 = 2$ and for $n \geq 1$, p_{n+1} is the largest prime factor of $p_1 p_2 \cdots p_n + 1$. Prove that $p_n \neq 5$ for any n.

Walkthrough —

(a) Show that $p_1 p_2 p_3 \cdots p_n + 1$ is odd for any $n \geq 1$, and p_n is odd for any $n \geq 2$. Deduce that $p_1 p_2 p_3 \dots p_n + 1$ is not a multiple of 3. (If you are stuck, then does verifying this statement for small values of n help?)

(b) What can be said about the smallest prime divisor of $p_1 p_2 p_3 \dots p_n + 1$?

(c) If it is a power of 5, then $p_1 p_2 p_3 \dots p_n$ is divisible by 4. Arrive at a contradiction.

Solution 8. Note that $p_1 p_2 \dots p_n + 1$ is odd for any $n \geq 1$, and hence p_n is odd for any $n \geq 2$. Since $p_1 = 2$ and $p_2 = 3$, it follows that for any $n \geq 2$, the integer $p_1 p_2 \dots p_n + 1$ is not divisible by any one of 2 and 3. So the least prime divisor of $p_1 p_2 \dots p_n + 1$ is at least 5 for any $n \geq 2$. If possible, suppose 5 is the largest prime divisor of $p_1 p_2 \dots p_n + 1$ for some integer $n \geq 2$. This yields

$$p_1 p_2 \dots p_n + 1 = 5^k$$

for some $k \geq 1$. This implies that 4 divides $p_1 p_2 \cdots p_n$, which is impossible since $p_1 = 2$, and p_r is odd for any integer $r \geq 2$. This shows that p_{n+1} is not equal to 5 for any integer $n \geq 2$. Consequently, it follows that $p_n \neq 5$ for any integer $n \geq 1$. ■

Example 1.13 (India RMO 2005 P2)**.** If x, y are integers and 17 divides both the expressions $x^2 - 2xy + y^2 - 5x + 7y$ and $x^2 - 3xy + 2y^2 + x - y$, then prove that 17 divides $xy - 12x + 15y$.

Walkthrough —

(a) Factorize $x^2 - 3xy + 2y^2 + x - y$ to show that

$$x \equiv y \pmod{17}, \quad \text{or } x \equiv 2y - 1 \pmod{17}$$

holds.

(b) Consider the above cases seperately, and use the divisibility of the other expression by 17 to obtain some congruence conditions on y. Using these conditions to read $xy - 12x + 15y$ modulo 17.

Solution 9. Let x, y be integers such that 17 divides both the expressions $x^2 - 2xy + y^2 - 5x + 7y$ and $x^2 - 3xy + 2y^2 + x - y$. Note that

$$x^2 - 3xy + 2y^2 + x - y = (x - y)(x - 2y + 1),$$

which is divisible by 17. It follows that

$$x \equiv y \bmod 17, \quad \text{or } x \equiv 2y - 1 \bmod 17$$

holds.

Let us consider the case that $x \equiv y \bmod 17$. It follows that

$$x^2 - 2xy + y^2 - 5x + 7y \equiv (x - y)^2 - 5x + 7y \equiv 2y \bmod 17.$$

Since 17 divides $x^2 - 2xy + y^2 - 5x + 7y$, we get $2y \equiv 0 \bmod 17$, which yields $x \equiv y \equiv 0 \bmod 17$, and hence 17 divides $xy - 12x + 15y$.

Let us consider the case that $x \equiv 2y - 1 \bmod 17$. Using $x^2 - 2xy + y^2 - 5x + 7y \equiv 0 \bmod 17$, we obtain

$$(2y - 1)^2 - 2(2y - 1)y + y^2 - 5(2y - 1) + 7y \equiv 0 \bmod 17,$$

which yields $y^2 - 5y + 6 \equiv 0 \bmod 7$. This implies that $(y-2)(y-3) \equiv 0 \bmod 17$. This shows that either $x \equiv 3 \bmod 17, y \equiv 2 \bmod 17$ holds, or $x \equiv 5 \bmod 17, y \equiv 3 \bmod 17$ holds. If $x \equiv 3 \bmod 17, y \equiv 2 \bmod 17$ holds, then

$$xy - 12x + 15y \equiv 6 - 36 + 30 \equiv 0 \bmod 17$$

holds. If $x \equiv 5 \bmod 17, y \equiv 3 \bmod 17$ holds, then we obtain

$$xy - 12x + 15y \equiv 15 - 60 + 45 \equiv 0 \bmod 17.$$

This proves that 17 divides $xy - 12x + 15y$. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)

[**Tao06**] Terence Tao. *Solving mathematical problems*. A personal perspective. Oxford University Press, Oxford, 2006, pp. xii+103. ISBN: 978-0-19-920560-8; 0-19-920560-4 (cited p. 7)