MOPSS

23 August 2025

Mathematics Olympiad

Problem Solving Sessions

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [Che25, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1

Exercise 1.1 (IOQM 2024 P8, AoPS)**.** Let n be the smallest integer such that the sum of the digits of n is divisible by 5, as well as the sum of the digits of $n+1$ is divisible by 5. What are the first two digits of n in the same order?

Summary — Consider the last digit of n. Next consider its second last digit, the third last digit etc.

Walkthrough —

(a) Show that the last digit of n is 9.

(b) Note that n has at least two digits. If n is a two-digit number, then show that it is not equal to any integer other than 19 or 69.

(c) Conclude that n has at least three digits, and its second last digit is 9.

(d) If n is a three-digit number, then show that it is not equal to any integer other than 299 or 799.

(e) Argue similarly to show that n has at least four digits, and its third last digit is 9.

(f) Prove that n has at least five digits, and its fourth last digit is 9.

(g) If n is a five-digit number, then show that it is equal to 49999.

Do we need to check that 49999 has the required property?

Exercise 1.2 (IOQM 2024 P10, AoPS)**.** Determine the number of positive integral values of p for which there exists a triangle with sides a, b and c which satisfy

$$a^2 + (p^2 + 9)b^2 + 9c^2 - 6ab - 6pbc = 0$$

Summary — Complete squares and use the triangle inequality.

Walkthrough —

(a) Note that

$$a^2 + (p^2 + 9)b^2 + 9c^2 - 6ab - 6pbc = (a - 3b)^2 + (pb - 3c)^2.$$

(b) This gives $a = 3b$ and

$$c = \frac{p}{3}b.$$

(c) Use the inequality[a] $a + b > c$, to obtain

$$3b + b > \frac{p}{3}b,$$

which yields $p \leq 11$.

(d) Use the inequality[b] $b + c > a$, to get

$$b + \frac{p}{3}b > 3b,$$

implying $p \geq 7$.

(e) It reduces to finding the number of positive integers p such that $7 \leq p \leq 11$.

What about the condition $c + a > b$? Anything else to be taken care of?

[a] How does it follow?
[b] How does it follow?

Exercise 1.3 (IOQM 2024 P15, AoPS)**.** Let X be the set of consisting twenty positive integers $n, n+2, \ldots, n+38$. The smallest value of n for which any three numbers $a, b, c \in X$, not necessarily distinct, form the sides of an acute-angled triangle is:

Summary — Use Pythagoras theorem.

Walkthrough —

(a) Use the fact that[a] if a, b, c are the sides of an acute-angled triangle with c denoting the largest side, then $a^2 + b^2 > c^2$ holds.

(b) Show that for any such n, the inequality

$$(n + 38)^2 < 2n^2$$

holds, which yields $n \geq 92$.

(c) Prove that for $n = 92$, the inequality

$$2(n + 2i)^2 > (n + 2i + 2)^2$$

holds for $i = 0, 1, \ldots, 18$.

(d) Note that if suffices to show that

$$2m^2 > (m+2)^2$$

for any $m \in \{92, 94, \ldots, 128\}$.

What is the role of part (c)? Does it guarantee that for $n = 92$, the set X has the stated property?

[a] How to prove it? Does the **converse** of this statement hold? In other words, if a, b, c are positive real numbers satisfying $a^2 + b^2 > c^2$, then does it follow that a, b, c are the sides of an acute-angled triangle?

Exercise 1.4 (INMO 2025 P5, AoPS, proposed by Pranjal Srivastava and Rohan Goyal)**.** Greedy goblin Griphook has a regular 2000-gon, whose every vertex has a single coin. In a move, he chooses a vertex, removes one coin each from the two adjacent vertices, and adds one coin to the chosen vertex, keeping the remaining coin for himself. He can only make such a move if both adjacent vertices have at least one coin. Griphook stops only when he cannot make any more moves. What is the maximum and minimum number of coins that he could have collected?

Walkthrough —

(a)

Exercise 1.5 (USAJMO 2013 P1, proposed by Titu Andreescu)**.** Are there integers a, b such that $a^5b + 3$ and $ab^5 + 3$ are both perfect cubes of integers?

Walkthrough —

(a) If 3 divides ab, then consider one of the integers $a^5b + 3, ab^5 + 3$ modulo 9.

(b) If 3 does not divide ab, then assuming the integers $a^5b + 3, ab^5 + 3$ to be perfect cubes, determine the integers a^5b, ab^5 modulo 9.

Exercise 1.6 (USAMO 2000 P4)**.** Find the smallest positive integer n such that if n squares of a 1000×1000 chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board.

Walkthrough —

(a) Does Fig. 1 help?

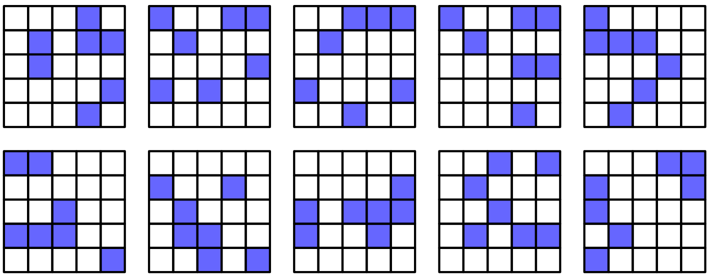
(b) Consider a suitable coloring to show that $n \geq 1999$.

(c) Does coloring 1999 squares suffice?

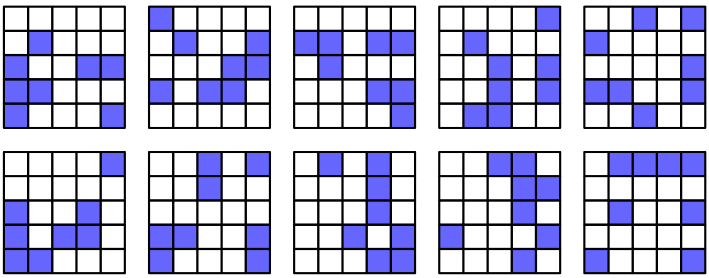
(d) What happens when there are two columns, each containing at least two colored squares, and some row contains one colored square from each of these columns?

(e) Consider the columns which contain at least two colored squares. How many colored squares can they contain together?

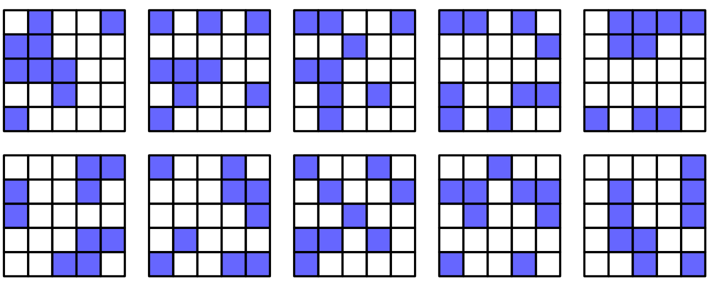
(f) What about the remaining colored squares? How many are they in total?



Each of the above chessboards has 7 of its squares colored



Each of the above chessboards has 8 of its squares colored



Each of the above chessboards has 9 of its squares colored

Figure 1: USAMO 2000 P4, Exercise 1.6

Exercise 1.7 (USAMO 2018 P4, proposed by Ankan Bhattacharya)**.** Let p be a prime, and let $a_1, a_2, \ldots, a_p$ be integers. Show that there exists an integer k

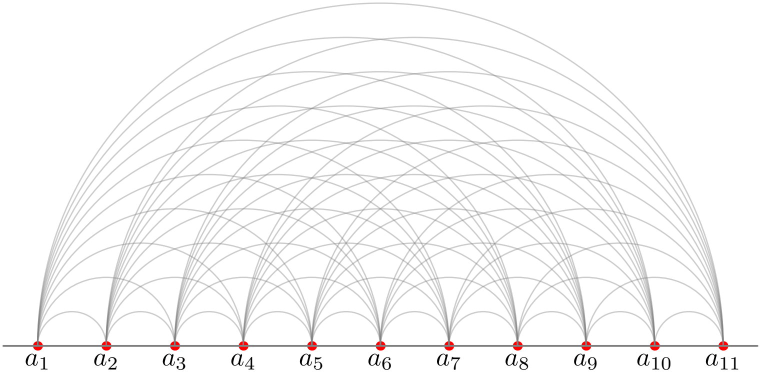


Figure 2: USAMO 2018 P4, Exercise 1.7

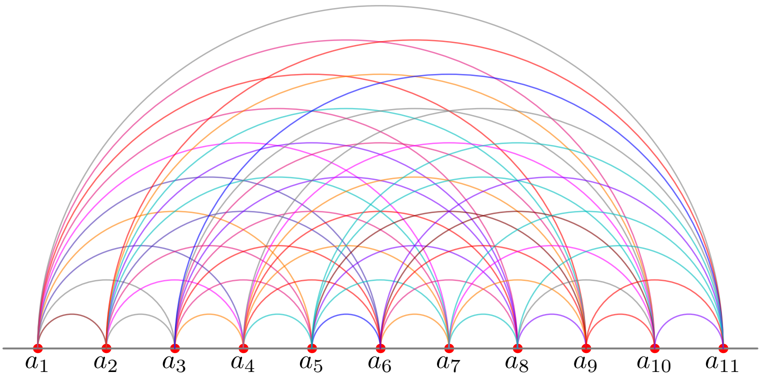


Figure 3: USAMO 2018 P4, Exercise 1.7

such that the numbers

$$a_1 + k, a_2 + 2k, \ldots, a_p + pk$$

produce at least $p/2$ distinct remainders upon division by p.

Walkthrough —

(a) Note that it suffices to show that some integer k among $1, 2, \ldots, p$ has the required property.

(b) Consider the integers $a_1, a_2, \ldots, a_p$, and we join each of them using an arc as in Fig. 2.

(c) For any two distinct integers i, j among $1, 2, \ldots, p$, show that the i-th and the j-th ones among the numbers

$$a_1 + k, a_2 + 2k, \ldots, a_p + pk,$$

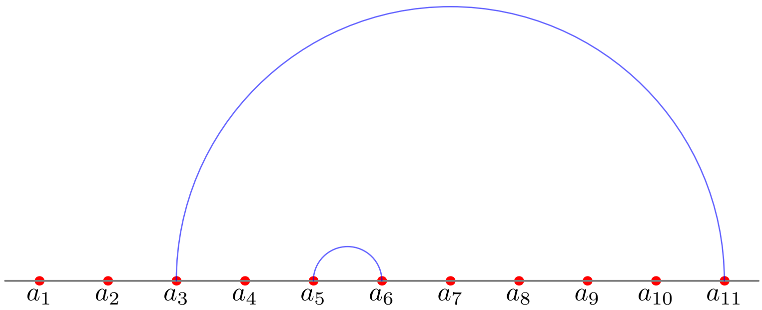


Figure 4: USAMO 2018 P4, Exercise 1.7

that is, the numbers $a_i + ik, a_j + jk$ leave the same remainder upon division by p precisely for one value of k.

(d) Let us color the arcs (see Fig. 2). We join a_i, a_j by an arc if the numbers $a_i + ik, a_j + jk$ leave the same remainder upon division by p. We write k on that arc. See Fig. 3, where instead of writing the corresponding integers on the arcs, we have colored them. Thinking of the integers $1, 2, \ldots, p$ as p colors, and we have *colored* the arcs instead of *labelling* the arcs[a].

(e) Note that there are precisely $\frac{p(p-1)}{2}$ unordered pairs of the form $\{a_i, a_j\}$, that is, there are precisely $\frac{p(p-1)}{2}$ arcs (see Fig. 2). Moreover, on each of these arcs, one of the integers $1, 2, \ldots, p$ is written. In Fig. 3, these arcs are colored using at most p colors.

(f) Conclude that for some integer k among $1, 2, \ldots, p$, at most $(p-1)/2$ arcs have the integer k written on them, that is, there are at most $(p-1)/2$ unordered pairs of the form $\{a_i, a_j\}$ such that $a_i + ik, a_j + jk$ produce the same remainder upon division by p. In Fig. 3, for some color, there are at most $(p-1)/2$ arcs of that color.

(g) Show that[b] for this integer k, the numbers

$$a_1 + k, a_2 + 2k, \ldots, a_p + pk$$

produce at least $p/2$ distinct remainders upon division by p.

[a] Observe that all the p colors might not have been used, for instance, if any two of $a_1, \ldots, a_p$ differ by a multiple of p.

[b] A graph on n vertices with m edges has at least $n - m$ connected components. This can be proved by induction on m, and observing that introducing a new edge reduces the number of connected components at most by 1.

References

[Che25] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)