MOPSS

9 August 2025

Mathematics Olympiad
Problem Solving Sessions

MOPSS

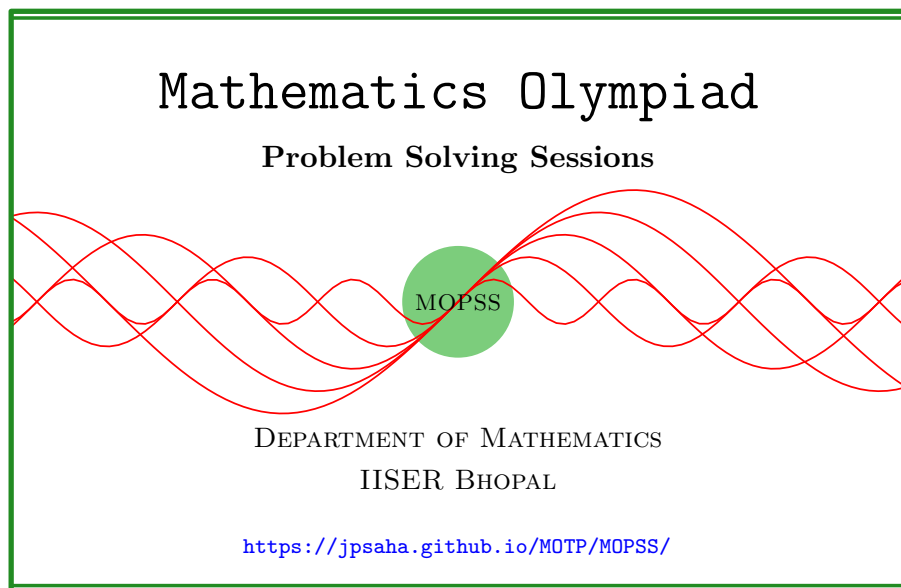DEPARTMENT OF MATHEMATICS
IISER BHOPAL

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at `https://blog.evanchen.cc/2017/03/06/on-reading-solutions/`.

- Evan Chen's Advice for writing proofs/Remarks on English, available at `https://web.evanchen.cc/handouts/english/english.pdf`.

- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].

- Tips for writing up solutions by Edward Barbeau, available at `https://www.math.utoronto.ca/barbeau/writingup.pdf`.

- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at `https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/`.

List of problems and examples

§1 Part A

Exercise 1.1 (IOQM 2023 P4, AoPS)**.** Let x, y be positive integers such that

$$x^4 = (x - 1)(y^3 - 23) - 1.$$

Find the maximum possible value of $x + y$.

> **Summary** — Look for some divisibility conditions.

> **Walkthrough** —
>
> **(a)** Note that $x \neq 1$ and $x - 1$ divides 2. It follows that $x = 2$ or $x = 3$. If $x = 2$, then $y^3 = 23 + 2^4 + 1 = 40$, which is impossible since y is an integer.
>
> **(b)** If $x = 3$, then $y^3 = 23 + \frac{1}{2}(3^4 + 1) = 23 + 41 = 64$, which gives $y = 4$.
>
> **(c)** It follows that $x + y = 7$.

Solution 1. Let x, y be positive integers satisfying the given equation. Note that $x \geq 2$. Using the congruence $x \equiv 1 \pmod{x - 1}$, it follows that $x - 1$ divides 2. This shows that $x = 2$ or $x = 3$. If $x = 2$, then we obtain

$$y^3 = 23 + 2^4 + 1 = 40,$$

which is impossible since y is an integer. If $x = 3$, then we obtain

$$y^3 = 23 + \frac{1}{2}(3^4 + 1) = 23 + 41 = 64,$$

which gives $y = 4$. Also note that $(3, 4)$ satisfies the given equation. Hence, it is the only solution over the positive integers. Thus, the maximum possible value of $x + y$ is $3 + 4 = 7$. ∎

Exercise 1.2 (IOQM 2023 P6, AoPS)**.** Let X be the set of all even positive integers n such that the measure of the angle subtended by a side at the center of some regular polygon is n degrees. Find the number of elements in X.

Summary — Determine the positive integers $r \geq 3$ such that dividing 360 by r yields an even number.

Walkthrough —

(a) Let $r \geq 3$ be an integer and consider a regular r-gon. Note that $360/r$ is even if and only if r is equal to one of

$$3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 24, 36, 45, 60, 120, 360.$$

In other words, there are precisely 16 positive integers $r \geq 3$ such that dividing 360 by r yields an even number.

(b) It follows that the number of elements of X is equal to 16.

Solution 2. Note that the map

$$r \mapsto \frac{360}{r}$$

is a bijection from the set of positive integers $r \geq 3$ such that $360/r$ is even to the set X, consisting of the even positive integers n such that the measure of the angle subtended by a side at the center of some regular polygon is n degrees. Since there are precisely 16 positive integers $r \geq 3$ such that dividing 360 by r yields an even number, it follows that the number of elements in X is 16. ∎

Exercise 1.3 (IOQM 2023 P9, AoPS)**.** Find the number of triples (a, b, c) of positive integers such that

(a) ab is a prime,

(b) bc is a product of two primes,

(c) abc is not divisible by square of any prime and

(d) $abc \leq 30$.

Walkthrough —

(a) Suppose (a, b, c) is a pair satisfying the above conditions.

(b) Since ab is a prime, it follows that one of a, b is equal to 1, and the remaining one is a prime.

(c) If $a = 1$, then b, c are distinct primes.

(d) If $b = 1$, then c is the product of two distinct primes which are not equal to a.

(e) This shows that (a, b, c) is equal to one of

$$(1, 2, 3), (1, 2, 5), (1, 2, 7), (1, 2, 11), (1, 2, 13),$$
$$(1, 3, 2), (1, 3, 5), (1, 3, 7),$$
$$(1, 5, 2), (1, 5, 3),$$
$$(1, 7, 2), (1, 7, 3),$$
$$(1, 11, 2),$$
$$(1, 13, 2),$$
$$(2, 1, 3 \cdot 5), (3, 1, 2 \cdot 5), (5, 1, 2 \cdot 3).$$

(f) Hence, the number of triples satisfying the above conditions is at most 17.

(g) Since any of the above triples does satisfy the given conditions, it follows that the number of triples satisfying the given conditions is equal to 17.

Exercise 1.4 (IOQM 2023 P10, AoPS)**.** The sequence $\langle a_n \rangle_{n \geq 0}$ is defined by $a_0 = 1, a_1 = -4$ and

$$a_{n+2} = -4a_{n+1} - 7a_n$$

for $n \geq 0$. Find the number of positive integer divisors of $a_{50}^2 - a_{49}a_{51}$.

Summary —— Compute a_n for small value of n, and then evaluate $a_n^2 - a_{n-1}a_{n+1}$ for small values of n. This may suggest a suitable pattern for $a_n^2 - a_{n-1}a_{n+1}$, which can be established by induction. Finally, compute $a_{50}^2 - a_{49}a_{51}$ and find its number of positive integer divisors.

Walkthrough ——

(a) Compute the first few terms of the sequence to note that

$$a_0 = 1,$$
$$a_1 = -4,$$
$$a_2 = 9,$$
$$a_3 = -8,$$
$$a_4 = -31,$$

which gives

$$a_1^2 - a_0a_2 = 16 - 9$$
$$= 7,$$
$$a_2^2 - a_1a_3 = 81 - 32$$

　　　　The content posted here and at this blog by Evan Chen are quite useful.

$$= 49,$$
$$a_3^2 - a_2 a_4 = 64 + 279$$
$$= 343.$$

This may suggest that

$$a_n^2 - a_{n-1} a_{n+1} = 7^n$$

holds for any integer $n \geq 1$.

Solution 3. Let us establish the claim below.

Claim — For any integer $n \geq 1$,

$$a_n^2 - a_{n-1} a_{n+1} = 7^n.$$

Proof of the Claim. Note that it holds for $n = 1$. Assume that $n \geq 1$ is a positive integer such that

$$a_n^2 - a_{n-1} a_{n+1} = 7^n$$

holds. Observe that

$$
\begin{aligned}
a_{n+1}^2 - a_n a_{n+2} &= a_{n+1}^2 - a_n(-4a_{n+1} - 7a_n) \\
&= a_{n+1}^2 + 4a_n a_{n+1} + 7a_n^2 \\
&= a_{n+1}^2 + 4a_n a_{n+1} + 7a_{n-1} a_{n+1} + 7^{n+1} \\
&= a_{n+1}(a_{n+1} + 4a_n + 7a_{n-1}) + 7^{n+1} \\
&= 7^{n+1}.
\end{aligned}
$$

By induction, it follows that This proves that

$$a_n^2 - a_{n-1} a_{n+1} = 7^n$$

holds for any integer $n \geq 1$. $\qquad \square$

Consequently,
$$a_{50}^2 - a_{49} a_{51} = 7^{50},$$
whose number of positive integer divisors is equal to 51. $\qquad \blacksquare$

Exercise 1.5 (IOQM 2023 P11, AoPS)**.** A positive integer m has the property that m^2 is expressible in the form $4n^2 - 5n + 16$ where n is an integer (of any sign). Find the maximum possible value of $|m - n|$.

Summary — Write the equation $m^2 = 4n^2 - 5n + 16$ as a quadratic polynomial in n. Note that its discriminant is a perfect square. Obtain a suitable factorization.

Walkthrough —

(a) Let m, n be integers satisfying $m^2 = 4n^2 - 5n + 16$, or equivalently,

$$4n^2 - 5n + (16 - m^2) = 0.$$

(b) Since n is an integer, the discriminant of the quadratic polynomial $4n^2 - 5n + (16 - m^2)$ in n is a perfect square, that is, there exists a nonnegative integer ℓ such that

$$25 - 16(16 - m^2) = \ell^2,$$

or equivalently, $16m^2 - \ell^2 = 231$, which gives

$$(4m - \ell)(4m + \ell) = 3 \cdot 7 \cdot 11.$$

(c) Note that $4m - \ell, 4m + \ell$ are odd and hence congruent to $1, -1$ modulo 4 (in some order). Also note that $4m - \ell < 4m + \ell$ holds.

(d) This shows that $(4m - \ell, 4m + \ell)$ is equal to one of

$$(1, 3 \cdot 7 \cdot 11), (3, 77), (7, 33), (11, 21),$$

and consequently, $8m$ is equal to one of $232, 80, 40, 32$, which shows that m is equal to one of $29, 10, 5, 4$.

(e) If $m = 29$, then $4n^2 - 5n - 825 = 0$, which gives $n = 15$.

(f) If $m = 10$, then $4n^2 - 5n - 84 = 0$ holds, which gives $n = -4$.

(g) If $m = 5$, then $4n^2 - 5n - 9 = 0$ holds, implying $n = -1$.

(h) If $m = 4$, then $n = 0$.

(i) Conclude that the maximum possible value of $|m - n|$ is equal to 14.

Solution 4. Let m be a positive integer and n be an integer satisfying

$$m^2 = 4n^2 - 5n + 16,$$

or equivalently, the equation

$$4n^2 - 5n + (16 - m^2) = 0.$$

Since n is an integer, the discriminant of the quadratic polynomial $4x^2 - 5x + (16 - m^2)$ is a perfect square, that is, there exists a nonnegative integer ℓ such that

$$25 - 16(16 - m^2) = \ell^2,$$

or equivalently,

$$16m^2 - \ell^2 = 231$$

holds. This yields
$$(4m - \ell)(4m + \ell) = 3 \cdot 7 \cdot 11.$$

Note that ℓ is an odd integer, and hence $4m - \ell, 4m + \ell$ are odd and congruent to $1, -1$ modulo 4 (in some order). Using $4m - \ell < 4m + \ell$, we obtain $(4m - \ell, 4m + \ell)$ is equal to one of

$$(1, 3 \cdot 7 \cdot 11), (3, 77), (7, 33), (11, 21).$$

This shows that $8m$ is equal to one of $232, 80, 40, 32$, which gives that m is equal to one of $29, 10, 5, 4$.

If $m = 29$, then
$$4n^2 - 5n - 825 = 0$$

holds, which gives $n = 15$. If $m = 10$, then
$$4n^2 - 5n - 84 = 0$$

holds, which implies $n = -4$. If $m = 5$, then
$$4n^2 - 5n - 9 = 0$$

holds, which yields $n = -1$. If $m = 4$, then
$$4n^2 - 5n = 0$$

holds, which gives $n = 0$.

Note that the pairs

$$(29, 15), (10, -4), (5, -1), (4, 0)$$

also satisfy the given equation. Hence, the maximum possible value of $|m - n|$ is

$$\max\{29 - 15, 10 - (-4), 5 - (-1), 4 - 0\} = \max\{14, 14, 6, 4\} = 14.$$

∎

Exercise 1.6 (IOQM 2023 P12, AoPS)**.** Let $P(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are integers and c is odd. Let p_i be the value of $P(x)$ at $x = i$. Given that $p_1^3 + p_2^3 + p_3^3 = 3p_1 p_2 p_3$, find the value of $p_2 + 2p_1 - 3p_0$.

Summary — Using parity conditions, show that $p_1 + p_2 + p_3 \neq 0$. This gives $p_1 = p_2 = p_3$. This gives a system of equations in a and b, which can be solved to find a and b.

Walkthrough —

(a) Note that if $p_1 + p_2 + p_3 \neq 0$, then using the condition that p_1, p_2, p_3 are real, it follows that $p_1 = p_2 = p_3$.

(b) Note that

$$p_1 = 1 + a + b + c,$$
$$p_2 = 8 + 4a + 2b + c,$$
$$p_3 = 27 + 9a + 3b + c,$$

which gives that

$$p_1 + p_2 + p_3 = 36 + 14a + 6b + 3c.$$

Since c is an odd integer, it follows that[a] $p_1 + p_2 + p_3 \neq 0$. Consequently, $p_1 = p_2 = p_3$.

(c) This gives
$$3a + b = -7, 5a + b = -19,$$
and hence, $a = -6, b = 11$.

(d) This gives

$$p_2 + 2p_1 - 3p_0 = 10 + 6a + 4b + 3c - 3c$$
$$= 18.$$

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[a]Since c is odd, it follows that p_2 is odd. Note that $p_1 - p_3$ is even (Why? Does it follow that $f(1) - f(3)$ is divisible by 2 for any polynomial $f(x)$ with integer coefficients?), and hence, $p_1 + p_2 + p_3$ is odd.

Solution 5. Since $P(x)$ is a polynomial with integer coefficients, it follows that $p_1 - p_3$ is divisible by 2. Since c is odd, it follows that p_2 is odd. This shows that $p_1 + p_2 + p_3$ is odd, and hence, $p_1 + p_2 + p_3 \neq 0$. Using $p_1^3 + p_2^3 + p_3^3 = 3p_1p_2p_3$, we obtain

$$(p_1 + p_2 + p_3)(p_1^2 + p_2^2 + p_3^2 - p_1p_2 - p_2p_3 - p_3p_1) = 0.$$

This gives
$$p_1^2 + p_2^2 + p_3^2 - p_1p_2 - p_2p_3 - p_3p_1 = 0,$$

oe equivalently,

$$(p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_3 - p_1)^2 = 0,$$

which implies that $p_1 = p_2 = p_3$. This shows that

$$1 + a + b + c = 8 + 4a + 2b + c = 27 + 9a + 3b + c,$$
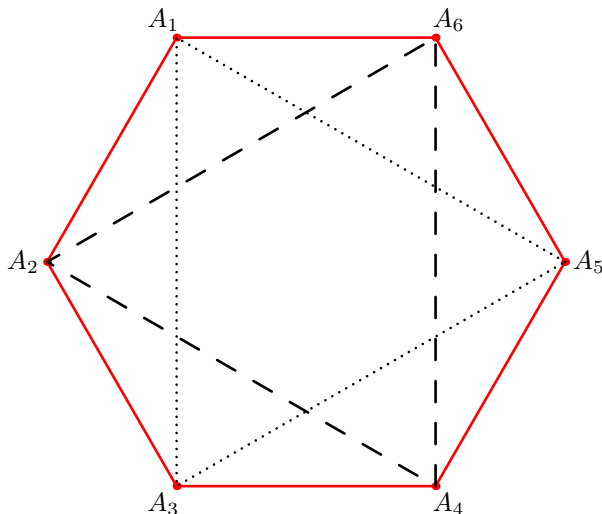
which yields

$$3a + b = -7, 5a + b = -19.$$

The content posted here and at this blog by Evan Chen are quite useful.

Figure 1: India IOQM 2023 P16, Exercise 1.7

This gives $a = -6, b = 11$. It follows that

$$p_2 + 2p_1 - 3p_0 = 10 + 6a + 4b + 3c - 3c$$
$$= 10 - 36 + 44$$
$$= 18.$$

∎

Exercise 1.7 (IOQM 2023 P16, AoPS, cf. ARML 2010 Team Problems P4).
The six sides of a convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored red. Each of
the diagonals of the hexagon is colored either red or blue. If N is the number
of such colorings such that every triangle $A_iA_jA_k$, where $1 \le i < j < k \le 6$,
has at least one red side, find the sum of the squares of the digits of N.

Summary — It suffices to (observe and then!!) show that for a coloring of the
diagonals of the hexagon using red and blue, if each of the triangles $A_1A_2A_3$
and $A_2A_4A_6$ has at least one red side, then each of the triangles $A_iA_jA_k$, where
$1 \le i < j < k \le 6$, has at least one red side too. Count the number of colorings
of the diagonals of the hexagon using red and blue such that each of the triangles
$A_1A_3A_5$ and $A_2A_4A_6$ has at least one red side.

Walkthrough —

   (a) The sides of the hexagon has been colored red.

   (b) Note that if $A_iA_jA_k$ is such that some two of its vertices are consecutive,

then it has at least one red side.

(c) If $A_iA_jA_k$ is such that no two of its vertices are consecutive, then any two of its vertices are exactly one vertex apart. Note that there precisely two such triangles, namely, $A_1A_3A_5$ and $A_2A_4A_6$.

(d) Observe that if each of these two triangles have at least one red side, then the required condition is satisfied.

(e) Moreover, if the required condition is satisfied, then each of these two triangles has at least one red side.

(f) It follows that N is equal to the number of colorings of the diagonals of the hexagon using red and blue such that each of the triangles $A_1A_2A_3$ and $A_2A_4A_6$ have at least one red side.

(g) This shows that

$$N = (2^3 - 1)(2^3 - 1)(2^{\binom{6}{2}-6-(3+3)}) = 7 \cdot 7 \cdot 8 = 392.$$

(h) The sum of the squares of the digits of N is equal to $3^2 + 9^2 + 2^2 = 9 + 81 + 4 = 94$.

Solution 6. Consider one coloring of the diagonals of the hexagon using red and blue. If this coloring satisfies the given condition, then each of the triangles $A_1A_2A_3$ and $A_2A_4A_6$ has at least one red side. Let us establish the claim below.

> **Claim —** For any coloring of the diagonals of the hexagon using red and blue, if each of the triangles $A_1A_2A_3$ and $A_2A_4A_6$ has at least one red side, then each of the triangles $A_iA_jA_k$, where $1 \le i < j < k \le 6$, has at least one red side.

Proof of the Claim. Consider a triangle $A_iA_jA_k$ with $1 \le i < j < k \le 6$. Note that if $A_iA_jA_k$ is such that some two of its vertices are consecutive, then it has at least one red side. If $A_iA_jA_k$ is such that no two of its vertices are consecutive, then any two of its vertices are exactly one vertex apart. Note that there precisely two such triangles, namely, $A_1A_3A_5$ and $A_2A_4A_6$. This proves the Claim. □

Hence, it follows that N is equal to the number of colorings of the diagonals of the hexagon using red and blue such that each of the triangles $A_1A_2A_3$ and $A_2A_4A_6$ have at least one red side. Note that the sides of $A_1A_2A_3$ can be colored in $2^3 - 1 = 7$ ways such that at least one of its sides is red. Similarly, the sides of $A_2A_4A_6$ can be colored in $2^3 - 1 = 7$ ways such that at least one of its sides is red. Note that there are precisely $\binom{6}{2} - 6 - 3 - 3$ diagonals in the hexagon which are not a side of these two triangles. Hence, the number of ways to color these 15 diagonals using red and blue satisfying the given condition is equal to

$$N = (2^3 - 1)(2^3 - 1)(2^{\binom{6}{2}-6-(3+3)}) = 7 \cdot 7 \cdot 2^{15} = 392.$$
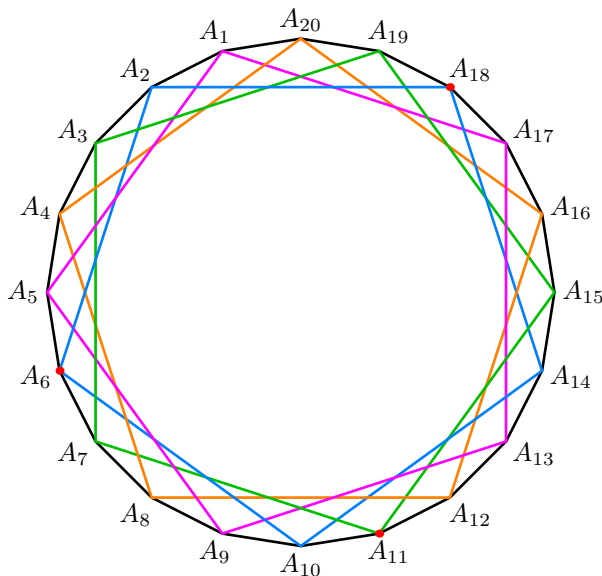

Figure 2: India RMO 2013, Exercise 1.8

The sum of the squares of the digits of N is equal to

$$3^2 + 9^2 + 2^2 = 9 + 81 + 4 = 94.$$

∎

Exercise 1.8 (RMO 2013d P6, AoPS)**.** Suppose that the vertices of a regular polygon of 20 sides are coloured with three colors red, blue and green, such that there are exactly three red vertices. Prove that there are three vertices A, B, C of the polygon having the same colour such that triangle ABC is isosceles.

> **Walkthrough** — Decompose the set of vertices of the 20-gon using the vertices of the pentagons as in Fig. 2. Apply the pigeonhole principle. It would be useful to note that any three vertices of a regular pentagon form the vertices of an isosceles triangle.

Solution 7. Since there are exactly three red vertices and any of the remaining 17 vertices are blue or green, it follows that at least 9 of these 17 vertices are of the same color, say blue. Note that the set of vertices of a regular 20-gon can be written as the union of the four pairwise disjoint sets, each of them consisting of the vertices of a regular pentagon (as in Fig. 2). Since there are nine blue vertices, by the pigeonhole principle, at least one of these four sets contains three blue points. Since any three points on a regular pentagon form the vertices of an isosceles triangle, the statement follows. ∎

Exercise 1.9 (RMO 2014a P4, AoPS). Is it possible to write the numbers $17, 18, 19, \ldots, 32$ in a 4×4 grid of unit squares with one number in each square such that if the grid is divided into four 2×2 subgrids of unit squares, then the product of numbers in each of the subgrids divisible by 16?
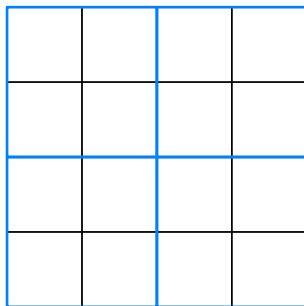

Figure 3: India RMO 2014, Exercise 1.9

Walkthrough —

(a) Note that the product of the entries in some subgrid is divisible by 32.

(b) Conclude that the product of all the 16 entries is divisible by $16 \times 16 \times 16 \times 32$.

(c) Is the product of the integers $17, 18, \ldots, 32$ divisible by $16 \times 16 \times 16 \times 32$?

Solution 8. The highest exponents of 2 dividing 32! and 16! are given by

$$v_2(32!) = 16 + 8 + 4 + 2 + 1, \quad v_2(16!) = 8 + 4 + 2 + 1.$$

So the highest power of 2 dividing the product of the integers $17, 18, 19, \ldots, 32$ is 2^{16}. Now if it were possible to write these numbers in a 4×4 grid in the above-mentioned manner, then the product of the numbers in each of the 2×2 subgrids with blue boundary (see Fig. 3) would be divisible by 2^4. Note that one such subgrid would contain 32, which implies that the product of $17, 18, \ldots, 32$ is divisible by $2^4 \cdot 2^4 \cdot 2^4 \cdot 32 = 2^{17}$, which is impossible. Hence it is not possible to write the integers $17, 18, \ldots, 32$ in a 4×4 grid satisfying the given conditions. ∎

Exercise 1.10 (RMO 2018a P4, AoPS). Let E denote the set of 25 points (m, n) in the xy-plane, where m, n are natural numbers, $1 \leq m \leq 5, 1 \leq n \leq 5$. Suppose the points of E are arbitrarily coloured using two colours, red and blue. Show that there always exist four points in the set E of the form $(a, b), (a + k, b), (a + k, b + k), (a, b + k)$ for some positive integer k such that at least three of these four points have the same colour. (That is, there always exist four points in the set E which form the vertices of a square and having at least three points of the same colour.)
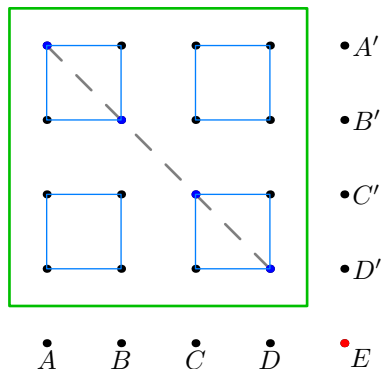
The content posted here and at this blog by Evan Chen are quite useful.

Figure 4: India RMO 2018, Exercise 1.10

> **Walkthrough** —
>
> **(a)** Assume that the conclusion is false, and by interchanging the colors (if necessary), assume that there are more red points than the blue ones.
>
> **(b)** Show that one of the four corners is red, and assume without loss of generality that the point E (as in Fig. 4) is red.
>
> **(c)** Considering the number of red points in each of the sets $\{A, A'\}$, $\{B, B'\}$, $\{C, C'\}$, $\{D, D'\}$, prove that these four sets together contain at most 4 red points, and conclude that the green square (as in Fig. 4) contains at least 8 red points.
>
> **(d)** Consider the blue squares in the green square, to show that the green square contains exactly 8 red points.
>
> **(e)** Conclude that there are exactly 13 red points, and each of the sets $\{A, A'\}$, $\{B, B'\}$, $\{C, C'\}$, $\{D, D'\}$ contains exactly one red point, and the green square contains exactly 8 red points.
>
> **(f)** For each point on the dashed diagonal (as in Fig. 4), consider the square having this point and E as the endpoints of one of its diagonals to show that the points on the dashed diagonal are blue.
>
> **(g)** Note that each point within the green square lying outside the dashed diagonal, is a vertex of a square having two of its remaining vertices on the dashed diagonal. Conclude that each such point is red.
>
> **(h)** Obtain a contradiction to conclude that the assmption is false.

Solution 9. On the contrary, let us assume that there is no axes-parallel square having at least three vertices of the same color.

Note that at least 13 among those 25 points are of the same color. Without loss of generality, assume that those are red. By our hypothesis, it follows that among the four vertices at the corners, there is at least one red vertex.

Without loss of generality[1], let us assume that the bottom-right vertex is red, and denote this vertex by E (as in Fig. 4).

Note that each of the sets $\{A, A'\}, \{B, B'\}, \{C, C'\}, \{D, D'\}$ (as in Fig. 4) contains at most one red point, otherwise, we can form an axes-parallel square with at least three vertices of the same color. Consequently, the green square (as in Fig. 4) contains at least $13 - 5 = 8$ red points.

If the green square contains at least 9 red points, then at least one of the four blue squares (as in Fig. 4) contains at least three red points, which is not the case by our hypothesis. Hence, the green square contains exactly 8 red points. Consequently, there are precisely 13 red points among the 25 points, and each of the sets $\{A, A'\}, \{B, B'\}, \{C, C'\}, \{D, D'\}$ contains exactly one red point.

For any given point on the dashed diagonal (as in Fig. 4), consider the square having this point and E as the endpoints of one of its diagonals. Note that one of the other two vertices of this square is red. By our hypothesis, it follows that all the points on the gray diagonal (as in Fig. 4) are blue.

Note that each point within the green square lying outside the dashed diagonal, is a vertex of a square having two of its remaining vertices on the dashed diagonal. This shows that all the points within the green square, lying outside the gray diagonal, are red. Hence, there exists an axes-parallel square, all whose vertices are red. This contradicts our hypothesis, and hence, it follows that there exists an axes-parallel square having at least three vertices of the same color. ∎

Exercise 1.11 (RMO 2023b P6, AoPS)**.** Consider a set of 16 points arranged in a 4×4 square grid formation. Prove that if any 7 of these points are coloured blue, then there exists an isosceles right-angled triangle whose vertices are all blue.

The following is from AoPS, is due to Rohan (Goyal?) as mentioned here by L567.

> **Walkthrough** —
> **(a)** Show that if the small square (as in Fig. 5a) does not contain a blue point, then we are done.
> **(b)** It remains to consider the case when the small square (as in Fig. 5a) contains at least one blue point.
> **(c)** Rotating the configuration about the center of the small square (if necessary), assume that the top-left vertex of the small square (as in Fig. 5a) is blue.
> **(d)** Prove that the gray square contains at most three blue points.
> - Consider the case when at least two blue points lie on the bigger dashed circle. Show that the smaller dashed circle does not contain any blue point, in this case. Hence, the gray square contains at

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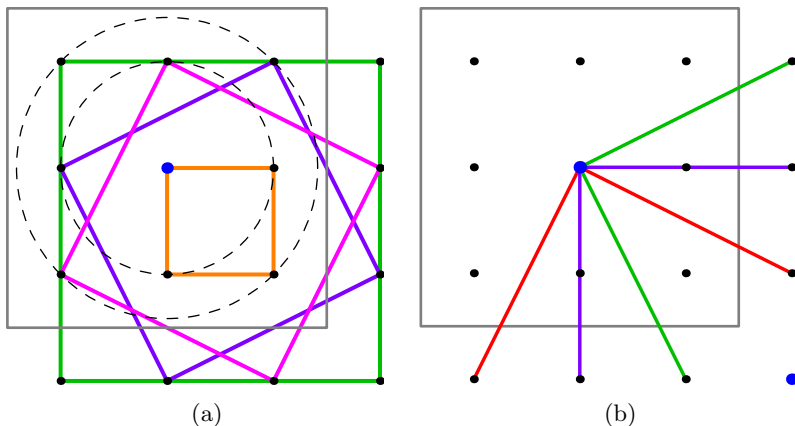[1]Convince yourself that there is **indeed no loss of generality** in assuming so.

     The content posted here and at this blog by Evan Chen are quite useful.

(a)   (b)

Figure 5: India RMO 2023, Exercise 1.11

> most three blue points.
>
> - Similarly, if the smaller dashed circle contains at least two blue points, then the gray square (as in Fig. 5b) contains at most three blue points.
>
> **(e)** It suffices to consider that each one of the red, purple, and green L-shapes, has at most one end-point which is blue (otherwise, we are done).
>
> **(f)** Use the above to show that the bottom-right point (as in Fig. 5b) is blue.
>
> **(g)** Consider the point at the bottom-right corner, and the center of the gray square, and a blue end-point of an L-shape, to show that these three points form the vertices of an isosceles triangle having the required properties.

Solution 10. Note that the 16 points are the vertices of the four squares as in Fig. 5a. If no vertex of the small square is blue, then by the pigeonhole principle, at least one of the remaining three squares has at least three blue vertices, and hence there exists an isosceles right-angled triangle with blue vertices.

Let us assume that the small square (as in Fig. 5a) has a blue vertex. Rotating the configuration about the center of the small square (if necessary), we may and do assume that the top-left vertex of the small square is blue. Henceforth, on the contrary, let us assume that there are no isosceles right-angled triangles with blue vertices.

> **Claim —** The gray square (as in Fig. 5a) contains at most three blue points.

Proof of the Claim. Note that the points within the gray square lies on the two dashed circles (as in Fig. 5a). Therefore, to prove the Claim, it suffices to

show that if one of the dashed circles (as in Fig. 5a) contains at least two blue points, then the other dashed circle does not contain any blue point.

If at least two blue points lie on the bigger dashed circle, then using the assumption, it follows that no more blue points lies on it, and hence these two blue points lie along a diameter. It follows that no blue point lies on the other dashed circle.

If at least two blue points lie on the smaller dashed circle, then using a similar argument, it follows that no blue point lies on the bigger dashed circle.

The Claim follows. □

Note that if both the end-points of one of the red, purple, and green L-shapes (as in Fig. 5b) are blue, then these points together with the center of gray square form the vertices of an isosceles right-angled triangle, contradicting the assumption. Hence, each one of these three L-shapes has at most one end-point which is blue. Since there are 7 blue points, using the Claim, it follows that the bottom-right point is blue. Note that the center of the gray square, the bottom-right point, and a blue end-point of the "purple" L-shape, form the vertices of a triangle having the required properties. ■

References

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The content posted here and at this blog by Evan Chen are quite useful.