Invariants and Colorings

Amit Kumar Mallik

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Problem 1 (Mimamsa 2019)**.** *There are* 1000 *glasses of wine,* 500 *each of red and blue. At each step, you choose two glasses and mix them up. Regardless of quantity, if you mix a red with a red, it becomes blue. Instead, if you mix a red with a blue, then it becomes red. Lastly, a blue with a blue gives blue. What will be the color of the last remaining glass?*

Problem 2. *The numbers* 1 *through* 2023*, inclusive, are written on a blackboard. A move consists of taking two written numbers* a *and* b*, erasing them, and writing* $ab + a + b$ *on the board. Continue this until only one number is left on the board. What is this number?*

Problem 3. *A pizza is cut into six slices. A pineapple is placed on first and third slice. You may add a pineapple each to two neighbouring slices. Is it possible to have equal number of pineapples on each slice?*

Problem 4. *Consider a* 8×8 *grid with the two diagonally opposite corners removed. Can you cover the remaining board with* 62 *squares using* 2×1 *or* 1×2 *dominoes?*

Problem 5. *Can a* 8×8 *grid be tiled with some* T*-tetrominoes and exactly one square tetromino?*

Problem 6 (Austria 2021 Regional)**.** *The numbers* $1, 2, ..., 2020$ *and* 2021 *are written on a blackboard. The following operation is executed: Two numbers are chosen, both are erased and replaced by the absolute value of their difference. This operation is repeated until there is only one number left on the blackboard. (a) Show that* 2021 *can be the final number on the blackboard. (b) Show that* 2020 *cannot be the final number on the blackboard.*

Problem 7 (Austrian 2023 Regional)**.** *Determine all natural numbers* $n \geq 2$ *with the property that there are two permutations* $(a_1, a_2, ..., a_n)$ *and* $(b_1, b_2, ..., b_n)$ *of the numbers* $1, 2, ..., n$ *such that* $(a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$ *are consecutive natural numbers.*

Problem 8. *You are given a* $m \times n$ *grid with an integer on each cell. In a move, you choose an integer and add the same to two adjacent cells. Determine for which initial board configurations is it possible to get zeros on each cell in finitely many moves.*

Suppose all the integers were non-negative to start with. Now, solve the same problem with the additional constraint that any integer in any intermediate board configuration is also non-negative.

Problem 9 (Canada MO 2024 P2)**.** *Jane writes down* 2024 *natural numbers around the perimeter of a circle. She wants the* 2024 *products of adjacent pairs of numbers to be exactly the set* $\{1!, 2!, \ldots, 2024!\}$*. Can she accomplish this?*

Problem 10 (PSS)**.** *On a board, a sequence of knight moves that starts at a any cell and goes over each cell exactly once (it may end anywhere), is a knight's path. A knight's path that ends at the starting cell is a knight's tour. For what values of* n*, does there exists a knight's tour on a* $4 \times n$ *board? What about knight's path?*

Problem 11 (Austria 2022 National)**.** *Each person stands on a whole number on the number line from* 0 *to* 2022 *. In each turn, two people are selected by a distance of at least* 2*. These go towards each other by* 1*. When no more such moves are possible, the process ends. Show that this process always ends after a finite number of moves, and determine all possible configurations where people can end up standing. (whereby is for each configuration is only of interest how many people stand at each number.)*

Problem 12. *Given* n *red points and* n *blue points on the plane in general position, show that one may pair them up such that the line segments joining them are disjoint.*

Problem 13. *Given a graph* G *on* n *vertices, show that we can color the vertices with the colors red and blue so that for every vertex, at least half of its neighbors are of opposite color.*

Problem 14 (Austria 2020 national)**.** *On a blackboard there are three positive integers. In each step the three numbers on the board are denoted as* a, b, c *such that* $a > gcd(b, c)$*, then* a *gets replaced by* $a - gcd(b, c)$*. The game ends if there is no way to denote the numbers such that* $a > gcd(b, c)$*.*

Prove that the game always ends and that the last three numbers on the blackboard only depend on the starting numbers.

Problem 15. 65 *beetles are placed on a* 9×9 *grid. Each minute, each beetle will move to an adjacent square, either horizontally or vertically, but never twice in the same orientation. So a beetle could move Left, Up, Left, Down, but could not go Left then Right. Prove that eventually two beetles will be on the same square.*

Problem 16 (INMO 2018 P5)**.** *There are* $n \geq 3$ *girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbours combined, the teacher takes away one apple from that girl and gives one apple each to her neighbours. Prove that, this process stops after a finite number of steps. (Assume that, the teacher has an abundant supply of apples.)*

Problem 17 (PSS)**.** *A candy is placed on each vertex of an* n*-gon. In one move, you may simultaneously move any two candies by one place in opposite directions The goal is to get all the candies into one vertex. For which* n *is the goal achievable.*

Problem 18 (Argentina OMA 2020)**.** *Let* n *be a positive integer. In each of* k *squares on an* $n \times n$ *board there is an infected person. Every day, everyone with at least two infected neighbors is simultaneously infected. Find the smallest possible value of* k *so that after a large enough number of days, all people are infected.*

Problem 19 (INMO 2020 P6)**.** *A stromino is a* 3×1 *rectangle. Show that a* 5×5 *board divided into twenty-five* 1×1 *squares cannot be covered by* 16 *strominos such that each stromino covers exactly three squares of the board, and every square is covered by one or two strominos. (A stromino can be placed either horizontally or vertically on the board.)*

Problem 20 (PSS). *Remove one corner of a $(2n+1) \times (2n+1)$ board. For what values of n can the rest of the board be covered with 2×1 and 1×2 dominoes such that there are exactly half of each type.*

Problem 21 (USAJMO 2023 P3). *Consider an n-by-n board of unit squares for some odd positive integer n. We say that a collection C of identical dominoes is a maximal grid-aligned configuration on the board if C consists of $(n^2-1)/2$ dominoes where each domino covers exactly two neighboring squares and the dominoes don't overlap: C then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let $k(C)$ be the number of distinct maximal grid-aligned configurations obtainable from C by repeatedly sliding dominoes. Find the maximum value of $k(C)$ as a function of n.*

Problem 22 (Canada MO 2018 P1). *Consider an arrangement of tokens in the plane, not necessarily at distinct points. We are allowed to apply a sequence of moves of the following kind: select a pair of tokens at points A and B and move both of them to the midpoint of A and B.*

We say that an arrangement of n tokens is collapsible if it is possible to end up with all n tokens at the same point after a finite number of moves. For what values of n, every arrangement of n tokens is collapsible.

Problem 23 (Belarus MO 2025). *A positive integer with three digits is written on the board. Each second the number n on the board gets replaced by $n + \frac{n}{p}$, where p is the largest prime divisor of n. Prove that either after 999 seconds or 1000 second the number on the board will be a power of two.*

Problem 24 (PSS). *Let P be any point in the interior of any even n-gon. Show that P is contained in an even number of triangles whose vertices are vertices of the n-gon.*

Problem 25 (PSS). *On one square of a 5×5 board, we write -1 and on the other 24 squares $+1$. In one move, you may reverse the signs of one $a \times a$ subsquare with $a > 1$. Your goal is to reach $+1$ on each square. On which squares should -1 be to reach the goal?*

Problem 26 (USAMO 2023 P4). *A positive integer a is selected, and some positive integers are written on a board. Alice and Bob play the following game. On Alice's turn, she must replace some integer n on the board with $n+a$, and on Bob's turn he must replace some even integer n on the board with $n/2$. Alice goes first and they alternate turns. If on his turn Bob has no valid moves, the game ends.*

After analyzing the integers on the board, Bob realizes that, regardless of what moves Alice makes, he will be able to force the game to end eventually. Show that, in fact, for this value of a and these integers on the board, the game is guaranteed to end regardless of Alice's or Bob's moves.

1 Hard Problems

Problem 27 (IMOSL 2017 C1). *A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.*

Problem 28 (IMOSL 2012 C1). *Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.*

Problem 29 (IMO 1993 P3). *On an infinite chessboard, a solitaire game is played as follows: at the start, we have n^2 pieces occupying a square of side n. The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed. For which n can the game end with only one piece remaining on the board?*

Problem 30 (USAMO 2023 P3). *Consider an n-by-n board of unit squares for some odd positive integer n. We say that a collection C of identical dominoes is a maximal grid-aligned configuration on the board if C consists of $(n^2 - 1)/2$ dominoes where each domino covers exactly two neighboring squares and the dominoes don't overlap: C then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let $k(C)$ be the number of distinct maximal grid-aligned configurations obtainable from C by repeatedly sliding dominoes. Find all possible values of $k(C)$ as a function of n.*

Problem 31 (IMOSL 2014 C2). *We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m .*