COMBINATORICS PROBLEMS

MAINAK

(1) In a classroom, the students have formed several cliques. Every clique consists of 5 students, and every student belongs to exactly 5 cliques. Is it possible to choose a student from every clique without choosing any student twice?

(2) Let $m \geq 15$ be a positive integer. Prove that, given a square piece of paper, one can cut it into m smaller squares (Not necessarily of the same size).

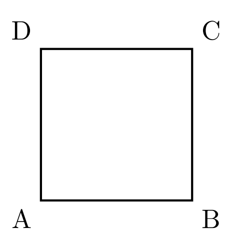
(3) Let S be a set consisting of n elements. In how many ways can one select an ordered triplet (A, B, C) of subsets of S such that $A \cup B \cup C = S$?

(4) Seven points are placed on a closed disc of radius 1 such that the distance between any two of these points is at least 1. Prove that one of the points is the center of the disc.

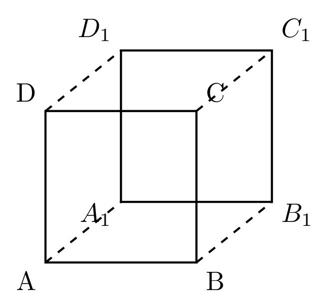
(5) Let G be a connected graph.
   (a) Show that if G is finite, then G contains a vertex v such that $G - v$ is connected.
   (b) Is this statement necessarily true if G is infinite?

(6) Define the k dimensional cube Q_k as follows:
- Q_0 is a graph with a single vertex and no edges.
- For all natural number $k \geq 1$, the graph Q_k is defined to be a graph constructed by taking two copies of Q_{k-1}, then joining every vertex from the first copy to the corresponding vertex in the second copy by an edge.
   (a) How many edges are there in the graph Q_k?
   (b) Prove that Q_k is a bipartite graph for all $k \in \mathbb{N}$.

Bonus: How many automorphisms does Q_k have?



Q_2



Q_3

(7) What is the minimum possible number a_n, such that the complete graph K_n can be written as the union of a_n many bipartite subgraphs?

(8) Let $G = [X, Y]$ be a simple bipartite graph where the partitions X and Y contains n vertices each, and every vertex in G has degreee at least $\frac{n}{2}$. Show that, G has a perfect matching.

(9) Four knights (two black and two white) are placed on a 3×3 chessboard, as in the **Figure 1** below. Is it possible to make a series of legal moves to change the configuration of the knights to that of **Figure 2**?

W		W
B		B

.

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Figure 1

W		B
B		W

Figure 2

(10) A pack of $m \times n$ cards with m values and n colors is made of one card of each value and color. The cards are arranged in an array with n rows and m columns. Show that one can pick n cards, one in each row, such that they all have different colors.

(11) (a) Let T be a tree. Prove that T has at most one perfect matching.
(b) Prove that T has a perfect matching if and only if for every vertex t of T, we have $o_T(\{t\}) = 1$.
For any set of vertices S, we define $o_T(S)$ to be the number of odd components in the graph obtained by deleting the vertices of S from T.

(12) The n^{th} *Catalan Number* C_n is the number of monotonic lattice paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal. A monotonic path is one which starts in the lower left corner, finishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards.
(a) Show (combinatorially) that $C_0 = 1$ and

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

for $n \geq 0$.
(b) Show that if $c(x) = \sum_{n \geq 0} C_n x^n$ is the generating function of C_n, then

$$c(x) = 1 + xc(x)^2.$$

(c) Show that

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

(13) Nine students take a test with 60 one-mark questions (with no negative or partial marking). Each of them got a positive score. Prove that one can always find two disjoint nonempty groups of students whose scores add up to the same number.
Bonus*: What if only 8 students took the test?

(14) Given a convex polygon with n sides where no three diagonals are concurrent, find the number of intersection points of the diagonals (inside the polygon).

(15) During a meeting, n-people are sitting around a round table. They all disperse for lunch break and come back after an hour to resume the meeting. In how many ways can they

sit around the table so that no one sits immediately to the left of someone they sat immediately to the right of before leaving for lunch?

Bonus*: In how many ways can they sit around the table so that no two persons sit next to each other both before and after the lunch?

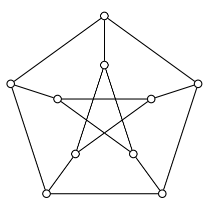
(16) Prove that for $n \geq 0$,

$$\sum_{k=0}^{\infty} \binom{n+k}{k} 2^{-(n+k+1)} = 1,$$

and

$$\sum_{k=0}^{n} \binom{n+k}{k} 2^{-(n+k+1)} = \frac{1}{2}.$$

(17) The following graph is called the Petersen graph



Does the Petersen graph have any Hamiltonian paths? How about any Hamiltonian cycles?

(18) How many 3×3 magic squares of weight k are there?

A magic square is an $n \times n$ array of numbers where every row and every column contains the numbers $1, 2, \ldots, n$ exactly once each.

Bonus: Let $g_n(k)$ be the number of $n \times n$ magic squares of weight k. Then prove that $g_n(k)$ is a polynomial in k with degree $(n-1)^2$.

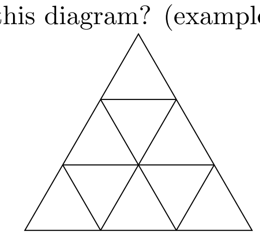
(19) Two square sheets of papers with side length 45 are both divided into 2025 polygonal regions of area 1 each by drawing lines on them (the divisions are not necessarily identical). We put the twp papers on top of each other. Prove that, one can place 2025 pins (with negligible cross section area) through these two papers so that all 4050 smaller regions get pierced.

(20) Let $A_1, A_2, \ldots, A_n$ be nonempty finite sets such that

$$\sum_{1 \leq i < j \leq n} \frac{|A_i \cap A_j|}{|A_i| \cdot |A_j|} < 1.$$

Show that the sets $A_1, A_2, \ldots, A_n$ have a system of distinct representatives.

(21) An equilateral triangle of side length n is divided into by n^2 equilateral triangles of side length 1 by drawing lines parallel to its sides. How many parallelograms are created in this diagram? (example below)



(As we can count, the answer is 15 for $n = 3$.)

(22) (a) Let $S = \{1, 2, \ldots, 200\}$ and T be a subset of S containing 100 elements such that no element of T divides another element of T. Prove that every element of T is greater than 15. (Recall that, if T contained 101 elements, then some element of T will divide another.)

(b) Construct such a set T containing 16.

(c) Given any $k \in \mathbb{N}$, if T a k-element subset of $\{1, 2, \cdots, 2k\}$ such that no element of T divides another, then find the minimum possible value of an element of T.

(23) Let G be a finite tournament, and let v and w be distinct G. Let M be the largest possible size of a set of pairwise edge disjoint directed paths from v to w. Similarly, let N be the largest possible size of a set of pairwise edge disjoint directed paths from w to v. Prove that $M = N$ if and only if v and w have the same outdegree.

(24) Recall that an FSU (finite simple undirected) graph G is called k-connected if G has more than k vertices and given any subset $T \subset V(G)$ with $|T| < k$, the graph $G \smallsetminus T$ is connected.

(a) Let G be a k-connected graph. Let H be the graph obtained by adding a vertex v to G and adding edges from v to k vertices of G. Prove that H is k-connected.

(b) Let G be an FSU graph. Prove that G is 2-connected if and only if for any three distinct vertices x, y, z there is a path in G from x to z containing y.