COMBINATORICS PROBLEMS

Part A

(1) A pizza parlor advertised that they have over a million possible pizza combinations. How many different toppings must they offer if their advertisement is true?

(2) In a classroom, the students have formed several cliques. Every clique consists of 5 students, and every student belongs to exactly 5 cliques. Prove that, the number of students is equal to the number of cliques.
Bonus: Is it possible to choose a student from every clique without choosing any student twice?

(3) Prove the following statements:
  (a) For all integers $n \geq 1$, one has $\sum_{i=1}^{n} i(i+1)(i+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$.
  Bonus: What will be a possible generalization?
  (b) For all integers $n \geq 1$, one has $\sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$.

(4) A palindrome is a word that is spelled the same way forward and backwards. As examples, take 'level' or 'refer'. How many n letter palindromes are there? (The words need not be meaningful)

(5) Let n be a positive integer. Prove that it is possible to cut up a cube into $7n+1$ smaller cubes. (Not necessarily of the same size)

(6) Let S be a set consisting of n elements.
  (a) In how many ways can one select an ordered pair (A, B) of subsets of S such that $A \cup B = S$?
  (b) In how many ways can one select an ordered pair (A, B) of subsets of S such that $A \cap B \neq \varnothing$?

(7) Seven points are placed on a closed disc of radius 1 such that the distance between any two of these points is at least 1. Prove that one of the points is the center of the disc.

(8) How many permutations of $1, 2, \ldots 9$ are there which have no odd number in the correct position?

(9) Let S be a set of natural numbers cardinality n. Prove that S contains a nonempty subset T such that the sum of elements of T is divisible by n.

(10) Let $A_1, A_2, \ldots A_k$ be subsets of a finite set A. For any element $b \in A$, define $n(b)$ to be the number of subsets A_i which contain b. Prove that

$$\sum_{i=1}^{k} |A_i| = \sum_{b \in A} n(b)$$

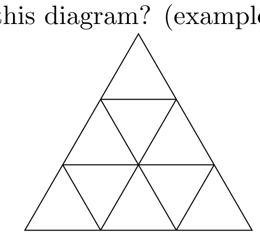
Part B

(1) Nine students take a test with 60 one-mark questions (with no negative or partial marking). Each of them got a positive score. Prove that one can always find two disjoint nonempty groups of students whose scores add up to the same number.
Bonus*: What if only 8 students took the test?

(2) Given a convex polygon with n sides where no three diagonals are concurrent, find the number of intersection points of the diagonals (inside the polygon).

(3) During a meeting, n-people are sitting around a round table. They all disperse for lunch break and come back after an hour to resume the meeting. In how many ways can they sit around the table so that no one sits immediately to the left of someone they sat immediately to the right of before leaving for lunch?
Bonus*: In how many ways can they sit around the table so that no two persons sit next to each other both before and after the lunch?

(4) An equilateral triangle of side length n is divided into by n^2 equilateral triangles of side length 1 by drawing lines parallel to its sides. How many parallelograms are created in this diagram? (example below)



(As we can count, the answer is 15 for $n = 3$.)

(5) Let $S = \{1, 2, \ldots, 200\}$ and T be a subset of S containing 100 elements such that no element of T divides another element of T. Prove that every element of T is greater than 15. (Recall that, we showed in class that if T contained 101 elements, then some element of T will divide another.)
Bonus 1: Construct such a set T containing 16.
Bonus 2: Given any $k \in \mathbb{N}$, if T a k-element subset of $\{1, 2, \cdots, 2k\}$ such that no element of T divides another, then find the minimum possible value of an element of T.