Orders and Primitive Roots Practice

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The first six problems were discussed in the lecture. Not all the problems involve explicit use of orders.

1. For any integer n such that $(n, 10) = 1$, find a relation between the order of 10 modulo n and the period of the decimal expansion of $\frac{1}{n}$. What if $(n, 10) > 1$?

2. Define the nth *Fermat number* F_n to be $2^{2^n} + 1$. Prove that if p is a prime which divides F_n, then $p \equiv 1 \pmod{2^{n+1}}$.

3. (IMO 2005/4) Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, \ n \geq 1.$$

4. Prove that for every odd positive number n, there is no integer m such that $n | m^{n-1} + 1$.

5. Let p be a prime. Prove that

$$1^k + 2^k + \ldots + (p-1)^k \equiv \begin{cases} 0 \pmod{p} & \text{if } p-1 \nmid k. \\ -1 \pmod{p} & \text{if } p-1 \mid k. \end{cases}$$

6. (IMO 1990/3) Determine all integers $n > 1$ such that $n^2 \mid 2^n + 1$.

7. Let $0 < x < y < z < p$ be integers where p is a prime. Prove that the following statements are equivalent:

   (a) $x^3 \equiv y^3 \pmod{p}$ and $x^3 \equiv z^3 \pmod{p}$

   (b) $y^2 \equiv zx \pmod{p}$ and $z^2 \equiv xy \pmod{p}$

8. (China TST 2006) Find all positive integer pairs (a, n) such that $n \mid (a+1)^n - a^n$.

9. Find all $n > 1$ such that $n \mid 2^n - 1$.

10. If $a > 1$ is an integer, prove that $n \mid \phi(a^n - 1)$.

11. The numbers m and n are positive integers, and the number p, $p > n$, is a prime. Given that $pm + n$ divides $p^p + 1$, prove that n divides m.

12. For $n \geq 2$, find all $n-$tuples of positive integers $(a_1, a_2, \cdots, a_n)$ such that

$$a_1 \mid 2^{a_2} - 1,\ a_2 \mid 2^{a_3} - 1, \cdots, a_{n-1} \mid 2^{a_n} - 1,\ a_n \mid 2^{a_1} - 1.$$

13. (Sweden TST 2022, modified) Let p be a prime such that $p^2 \mid 2^{p-1} - 1$. Prove that $p - 1$ has at least two distinct prime factors.

14. (Bulgaria 2006) Let p be a prime such that p^2 divides $2^{p-1} - 1$. Prove that for all positive integers n the number $(p-1)(p! + 2^n)$ has at least 3 different prime divisors.

15. (USA TST 2003) Find all ordered triples of primes (p, q, r) such that

$$p \mid q^r + 1, \quad q \mid r^p + 1, \quad r \mid p^q + 1.$$

16. (IMO Shortlist 2006 N2) Let x be a rational number such that $0 < x < 1$, let its decimal representation be $x = 0.x_1x_2x_3x_4\ldots$.

    Let $y \in (0, 1)$ be the number whose n-th digit after the decimal point is the 2^n-th digit after the decimal point of x, i.e. $y = 0.x_2x_4x_8x_{16}\ldots$. Show that if x is rational then so is y.

17. (China MO 2009) Find all the pairs of prime numbers (p, q) such that $pq \mid 5^p + 5^q$.

18. (IMO Shortlist 1998 N5) Determine all positive integers n for which there exists an integer m such that $2^n - 1$ is a divisor of $m^2 + 9$.

19. Determine all integers $n > 1$ such that $n^2 \mid 3^n + 1$.

20. One can prove better results than Problem 2:

(a) Define $F_n(a,b) = a^{2^n} + b^{2^n}$. Let p be a prime dividing $F_n(a,b)$ for some positive integers a, b with $\gcd(a,b) = 1$. Suppose $p = k \cdot 2^m + 1$ with k odd.

Let $u \equiv a/b \pmod p$. Suppose that u is a 2^t-th power residue mod p but not a 2^{t+1}-th power residue: this means that $u \equiv w^{2^t} \bmod p$ for some w, but $u \not\equiv w^{2^{t+1}} \pmod p$ for any w. (If u is not even a quadratic residue, take $t = 0$)

Then prove that $m = n + t + 1$.

(b) Using the fact that 2 is a quadratic residue modulo any prime which is $1 \bmod 8$, conclude that any prime p dividing $F_n = 2^{2^n} + 1$ is of the form $k \cdot 2^{n+2} + 1$.

(c) (Strengthened form of China TST 2005) Let n be a positive integer, and let $F_n = 2^{2^n} + 1$. Prove that for $n \geq 4$, there exists a prime factor of F_n which is larger than $2^{n+4}(n+1)$.

21. Let $p = k \cdot 2^n + 1$ be a prime, with $k \geq 3$ odd and $k \mid n$. Then

$$p \mid k^{2^m} + 1 \qquad \text{for some} \qquad m \leq n - 2.$$

22. (Bulgaria 2016) Find all positive integers m and n such that

$$mn \mid (2^{2^n} + 1)(2^{2^m} + 1).$$