INMOTC Karnataka Geometry 2024

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1. Points E and F are given on side BC of a convex quadrilateral ABCD (with E closer than F to B). Suppose angle EAB = angle CDF and angle FAE = angle FDE. Prove that angle CAF = angle EDB.
2. Two intersecting circles C1 and C2 have a common tangent which touches C1 at P and C2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. The line PN meets the circle C2 again at R. Prove that MQ bisects angle PMR.
3. In an acute-angled triangle ABC, AM is the median on BC, AL is the internal angle bisector of ∠BAC and AH is the altitude (H lies between L and B). It is known that ML = LH = HB. Find the ratios of the sidelengths of ABC.
4. I is the incentre of triangle ABC. Points X, Y are located on the line segments AB, AC respectively so that $BX.AB = IB^2$ and $CY.AC = IC^2$. Given that X, I, Y are collinear, find the possible values of the measure of angle A.
5. Two intersecting circles C_1 and C_2 have a common tangent intersecting C_1 in P and C_2 in Q. The 2 circles intersect in M and N where N is nearer to PQ than M. Prove that the triangles MNP and MNQ have equal areas.
6. A class of problems:
   (a) Let ABCD be a convex quadrilateral with diagonal point E. Let F, G, H, and I be the centroids of triangles △ABE, △BCE, △CDE, and △DAE, respectively. Then FGHI is a parallelogram
   (b) Let ABCD be a convex quadrilateral with diagonal point E. Let F, G, H, and I be the orthocentres of triangles △ABE, △BCE, △CDE, and △DAE, respectively. Then FGHI is a parallelogram.
   (c) Let ABCD be a convex quadrilateral with diagonal point E. Let F, G, H, and I be the circumcentres of triangles △ABE, △BCE, △CDE, and △DAE, respectively. Then FGHI is a parallelogram.
7. Let O, I be circumcentre and incentre of a triangle and let R, r be circumradius and inradius. Show $OI^2 = R^2 - 2Rr$.
8. Points K, L, M and N lying on the sides AB, BC, CD, and DA of a square ABCD are vertices of another square. Lines DK and NM meet at point E, and lines KC and LM meet at point F. Prove that EF is parallel to AB.
9. Let $ABCD$ be a convex quadrilateral with $\angle DAB = 90°$. Let M be the midpoint of BC. Given that $\angle ADC = \angle BAM$, prove that $\angle ADB = \angle CAM$.
10. Let l and m be lines through a given point A and a suppose a segment $\overline{AB}$ is given so that m is in between l and $\overline{AB}$. Locate a point C, using straight edge and compass, on the line l so that m is a median of triangle ABC.
11. Let I be the incentre and I_A be the A-excentre of a triangle ABC. Prove that the circumcircle of triangle ABC passes through the midpoint of the line segment II_A.
12. Let ABCD be a trapezoid with BC parallel to AD and AB = CD. A circle ω centred at I is tangent to the segments AB, CD, and DA. The circle BIC meets the side AB at points B and E. Prove that CE is tangent to ω.
13. Let ABC be an isosceles triangle with $AB = AC$. Suppose that the angle bisector of $\angle B$ meets AC at D and that $BC = BD + AD$. Determine $\angle A$.

14. Let A, B, P be three points on a circle. Prove that if a and b are the distances from P to the tangents at A and B and c is the distance from P to the chord AB, then $c^2 = ab$.

15. Prove that four circles circumscribed about four triangles formed by four intersecting straight lines in the plane have a common point.

16. The point O is situated inside the parallelogram ABCD so that $\angle AOB + \angle COD = 180°$. Prove that $\angle OBC = \angle ODC$.

17. $ABCD$ is a convex quadrilateral with $AD = BC$. Let E, F be midpoints of CD, AB respectively. Suppose rays AD, FE intersect at H and rays BC, FE intersect at G. Show that $\angle AHF = \angle BGF$.

18. Let ABC be an acute-angled triangle. The feet of the altitudes from A,B and C are D, E and F respectively. Prove that $DE + DF \leq BC$ and determine the triangles for which equality holds.

19. Given a triangle, show that the reflection of the orthocentre in a side of a triangle lies on the circumcircle of the triangle.

20. Let O be the center of the circumcircle ω of an acute-angle triangle ABC. A circle ω_1 with center K passes through A, O, C and intersects AB at M and BC at N. Point L is symmetric to K with respect to line NM. Prove that $BL \perp AC$.

21. Let a, b, c and d denote the sides of a cyclic quadrilateral and m and n its diagonals. Prove that $mn = ac + bd$.

22. Let BC be a chord of a circle Γ. Let ω be a circle tangent to chord BC at K and internally tangent to ω at T. Then ray TK passes through the midpoint M of the arc BC not containing T.

23. Let ABC be a triangle and AD,BE,CF be altitudes with H as orthocenter with D,E,F on sides. Suppose X is the intersection of circle passing through points A,E, F and the circumcircle of triangle ABC. Let EF meet BC at T. Show that A, T, X are collinear.

24. Let ABC be a triangle with orthocenter H and let D,E,F be the feet of the altitudes lying on the sides BC, CA, AB respectively. Let $T = EF \cap BC$. Prove that TH is perpendicular to the A-median of triangle ABC.

25. Let AB be the diameter of a circle Γ and let C be a point on Γ different from A and B. Let D be the foot of perpendicular from C on to AB.Let K be a point on the segment CD such that AC is equal to the semi perimeter of ADK.Show that the excircle of ADK opposite A is tangent to Γ.

26. Let E and F be the midpoints on the respective sides CA and AB of triangle ABC, and let P be the second point of intersection of the circles ABE and ACF . Prove that the circle AEF intersects the line AP again in the point X for which AX = 2XP .

27. Let the incircle of triangle ABC touch side BC at D, and let DT be a diameter of the circle. If line AT meets BC at X, then $BD = CX$.

28. Prove that the midpoints of the sides, feet of the altitudes, and the midpoints of the segments joining a vertex to the orthocentre lie on a common circle whose centre is the midpoint of the segment joining the circumcentre and the orthocentre and the diameter is the circumradius.Additionally, show that this circle is tangent to the incircle and the excircles!

29. Let ABCD be a square. Let P be point inside the square such that PA = 1,PB = 2,PC = 3. Find ∠APB in degrees.

30. ABCD is a unit square. Points P,Q,M,N are on sides AB, BC, CD, DA respectively such that AP + AN + CQ + CM = 2. Prove that PM⊥QN.

31. Show that the composition of two rotations of angle magnitudes a and b, respectively around centres A and B, is equal to a rotation a + b around another centre X. This centre X is located at a position where ∠XAB = a/2 and ∠ABX = b/2.

32. (INMO 2024)Let points A_1, A_2 and A_3 lie on the circle Γ in counter-clockwise order, and let P be a point in the same plane. For $i \in \{1, 2, 3\}$, let τ_i denote the counter-clockwise rotation of the plane centered at A_i, where the angle of the rotation is equal to the angle at vertex A_i in $\triangle A_1A_2A_3$. Further, define P_i to be the point $\tau_{i+2}(\tau_i(\tau_{i+1}(P)))$, where indices are taken modulo 3 (i.e., $\tau_4 = \tau_1$ and $\tau_5 = \tau_2$). Prove that the radius of the circumcircle of $\triangle P_1P_2P_3$ is at most the radius of Γ.

33. Prove that:

    (a) the bisectors of the exterior angles of a triangle intersect the extensions of its opposite sides at three points lying on the same straight line;

    (b) the tangents drawn from the vertices of the triangle to the circle circumscribed about it intersect its opposite sides at three collinear points.

34. In the plane let C be a circle, l a line tangent to the circle C and M a point on l. Find the locus of all points P with the following property: there exists two points Q, R on l such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.

35. Let M be the midpoint of the arc ACB on the circumcircle of $\triangle ABC$, and let MD be the perpendicular to the longer of AC and BC, say AC. We will call D as the C-Archimedes point.

    (a) Then show that D bisects the polygonal path ACB, that is,

    $$AD = DC + CB.$$

    (b) Let P be the midpoint of AB, then show that DP is parallel to the C-angle bisector of triangle ACB.

    (c) Suppose E, F are A-Archimedes point, B-Archimedes point of $\triangle ABC$, and Q, R are midpoints of BC and CA, respectively. Show that segments EQ, FR and DP concur. [Hint: look at the medial triangle.]

36. Let ABC be a triangle with $AB > AC$. Let P be a point on the line AB beyond A such that $AP+PC = AB$. Let M be the mid-point of BC and let Q be the point on the side AB such that $CQ \perp AM$. Prove that $BQ = 2AP$.

37. (Gauss-Newton line) A straight line intersects the sides AB, BC, and the extension of the side AC of a triangle ABC at points D, E, and F, respectively. Prove that the midpoints of the line segments DC, AE, and BF lie on a straight line.

38. Given a quadrilateral $ABCD$, let P, Q, R and S be resp. the orthocenters of triangles ABC, BCD, CDA and DAB. Prove that quadrilaterals $ABCD$ and $PQRS$ have the same area.[Hint: used signed areas.]

39. Prove that the perpendiculars drawn from the midpoints of the sides of a cyclic quadrilateral to the opposite sides are concurrent.

40. **Quadrilateral problems**:

    (a) In a quadrilateral ABCD, the circles inscribed in the triangles ABC, BCD, CDA, DAB are of the same radius. Prove that the given quadrilateral is a rectangle.

    (b) A convex quadrilateral is separated by its diagonals into four triangles. The circles inscribed in these triangles are of the same radius. Prove that the given quadrilateral is a rhombus.

    (c) The diagonals of a quadrilateral separate the latter into four triangles having equal perimeters. Prove that the quadrilateral is a rhombus.

41. Let AB be the hypotenuse of a right-angled triangle ABC. Let M be the midpoint of AB and O be the centre of circumcircle ω of triangle CMB. Line AC meets ω for the second time in point K. Segment KO meets the circumcircle of triangle ABC in the point L. Prove that the segments AL and KM meet on the circumcircle of triangle ACM.

42. Consider the parallelogram ABCD with obtuse angle A. Let H be the feet of perpendicular from A to the side BC. The median from C in triangle ABC meets the circumcircle of triangle ABC at the point K. Prove that points K, H, C, D are concyclic.

43. Two circles are internally tangent at N. The chords BA and BC of the larger circle are tangent to the smaller circle at K and M respectively. Q and P are midpoint of arcs AB and BC respectively. Circumcircles of triangles BQK and BPM are intersect at L. Show that BPLQ is a parallelogram.