Geometry-I

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§1 Warm up

We start with the orthocentre warmup. Let H be the orthocentre of triangle ABC with X, Y, Z projections of A, B, C respectively on opposite sides. Join AX, BY, CZ, XY, YZ, ZX. Note that quadrilateral $ZHXB$ is cyclic.

1. Show that H is the incentre of triangle XYZ. Mark all the angles possible.
2. What are the orthocentres of triangles HAB, HBC and HCA?
3. Let the AX extended meet circumcircle of $\triangle ABC$ at P. Show that quadrilateral $BHPC$ is a kite.
4. What is the circumradius of triangle HAB in terms of the circumradius of $\triangle ABC$?
5. Show that $|AH| = 2R\cos A$.

Next we look at the incentre/excentre warmup. Let I, I_A, I_B, I_C denote the incentre, A-excentre, B-excentre,C-excentre of triangle ABC. Let D, E, F be the meeting points of internal bisectors of angles A,B,C respectively on the opposite sides. Let T, T' be touchpoints of the incircle and A-excircle on BC respectively.

1. Show that $|BD| : |DC| = |AB| : |AC|$.
2. Show that $|BT| = s - b$ and $|BT'| = s - c$.
3. Compute $\angle BIC$ and $\angle BI_AC$.
4. Show that $|AI| = \dfrac{r}{\sin \dfrac{A}{2}}$.

§2 Few theorems

In triangle ABC with circumradius R and circumcentre O, let P, Q, S be the midpoint of sides AB, AC, BC. Let K, L, M be the midpoints of AH, BH, CH.

1. Show that $LMQP$ is a rectangle. Further show that L, S, M, Q, K, P are concyclic and call the circle ω. [Hint: Look at the centre of the three rectangles.]
2. Show that projections of the vertices on the opposite sides lie on the circle ω.
3. Show that $AOSK$ is a parallelogram. Conclude that the segment PS is congruent to the circumradius and thus the circumdiameter of ω is R.[Use $AH = 2R\cos A$.]

4. Show that $KOSH$ is a parallelogram and conclude that the centre of ω is midpoint of OH.

5. Let P be a point on the circumcircle. Show that the locus of the midpoint of HP is ω.

The circle ω is called the nine point circle of triangle ABC.

Let G be the centroid of $\triangle ABC$ and let N be the ninepoint centre of triangle ABC.

1. Extend the ray $\vec{OG}$ to X so that $|XG| : |GO| = 2 : 1$. Show that X is the orthocentre of triangle ABC. The line OH is called the *Euler line* of triangle ABC.

2. Note that N belongs to the Euler line of $\triangle ABC$.

Let I, I_A, I_B, I_C denote the incentre, A-excentre, B-excentre,C-excentre of triangle ABC.

1. Show that I is the orthocentre of $\triangle I_A I_B I_C$.

2. Show that the circumcircle of $\triangle ABC$ (denoted by ω) is the ninepoint circle of $\triangle I_A I_B I_C$.

3. Conclude that the midpoint of II_A is on ω and further the circle with II_A as diameter passes through B, C.

4. Show that the exterior angle bisector of B meets ω again at the midpoint of $I_A I_B$.

Given triangle ABC, construct equilateral triangles XBC, Y CA, and ZAB externally on the sides. Let the circumcircle of XBC intersect the line AX at F.

1. Prove that $\angle BFX = \angle XFC = 60°$.

2. Prove that A, F, C, Y are concyclic. Similarly, A, F, B, Z are also concyclic.

3. Prove that B, F, Y are collinear. Similarly, C, F, Z are also collinear. The point F is called the *Fermat point* of triangle ABC. It is the point of concurrency of the lines AX, BY, CZ. It is also the common point of the circumcircles of the equilateral triangles XBC, YCA, ZAB.

4. Let P be a point inside an acute $\triangle ABC$, show that the sum $|PA| + |PB| + |PC|$ is minimized when P equals the Fermat point. [Hint: Rotate the triangle PBC about B by 60° to P'BC'. Now look at CC'.]

§3 Exercises

1. Let X, Y be projections of B and C on opposite sides for an acute angled triangle ABC. Show that the midpoint of AH bisects the arc XY of the ninepoint circle.

2. Let M' be the midpoint of the arc BAC of the circumcircle of triangle ABC. Show that each of $M'BI_C$ and $M'CI_B$ is an isosceles triangle. Deduce that M' is indeed the midpoint of the segment $I_B I_C$.

3. The incircle of a triangle ABC is tangent to sides AB and AC at D and E respectively, and O is the circumcenter of triangle BCI. Prove that $\angle ODB = \angle OEC$.

4. Show that the exterior angle bisector of B meets circumcircle of $\triangle ABC$ again at the midpoint of $I_A I_B$.

5. Given an acute triangle ABC with a circumcircle ω, let the exterior angle bisector of angle B meet ω again at K. If M denotes the midpoint of the minor arc AC, then show that $\angle KCM$ is right.

6. Show that the Euler lines of triangles ABC, HAB, HBC, HCA are concurrent.

7. Show that the perpendiculars from A, B, C to the lines YZ, ZX, XY are concurrent.

8. Let P, Q be the projections of the vertex A on the angle bisectors of angle B. Show that the line PQ passes through the midpoint of the side AC. Conclude that the 4 projections of the vertex A on the angle bisectors of B and C are collinear.

9. Show that the line joining the projections of the orthocentre of a triangle on the angle bisectors of an angle of the triangle passes through the midpoint of the side opposite to the vertex considered. [Hint: It passes through the ninepoint centre.]

10. Let I be the incenter of a triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D. Let E be a point on the arc BDC and F a point on the side BC such that
$$\angle BAF = \angle CAE < \tfrac{1}{2}\angle BAC.$$
Finally, let G be the midpoint of $\overline{IF}$. Prove that $\overline{DG}$ and $\overline{EI}$ intersect on Γ. [Hint: See the problem from the point of view of I. Double every point from I. Do you notice that G goes to F and D to the A-excenter?]