Optimization problems

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1 Lecture problems

Problem 1 (INMO 2024). *All the squares of a 2024×2024 board are coloured white. In one move, Mohit can select one row or column whose every square is white, choose exactly 1000 squares in that row or column, and colour all of them red. Find maximum number of squares Mohit can colour in a finite number of moves.*

Problem 2 (Belarus MO 2024 P10.2). *Some vertices of a regular 2024-gon are marked such that for any regular polygon, all of whose vertices are vertices of the 2024-gon, at least one of its vertices is marked. Find the minimum possible number of marked vertices.*

Problem 3 (JBMOSL 2022 P1). *Anna and Bob, with Anna starting first, alternately color the integers of the set $S = \{1, 2, ..., 2022\}$ red or blue. At their turn each one can color any uncolored number of S they wish with any color they wish. The game ends when all numbers of S get colored. Let N be the number of pairs (a, b), where a and b are elements of S, such that a, b have the same color, and $b - a = 3$.*

Anna wishes to maximize N. What is the maximum value of N that she can achieve regardless of how Bob plays?

Problem 4 (Canada MO 2018 P1). *Consider an arrangement of tokens in the plane, not necessarily at distinct points. We are allowed to apply a sequence of moves of the following kind: select a pair of tokens at points A and B and move both of them to the midpoint of A and B.*

We say that an arrangement of n tokens is collapsible if it is possible to end up with all n tokens at the same point after a finite number of moves. For what values of n, every arrangement of n tokens is collapsible.

Problem 5 (Swiss Final Round 2023 P8). *Let n be a positive integer. We start with n piles of pebbles, each initially containing a single pebble. One can perform moves of the following form: choose two piles, take an equal number of pebbles from each pile and form a new pile out of these pebbles. Find (in terms of n) the smallest number of nonempty piles that one can obtain by performing a finite sequence of moves of this form.*

2 Additional Problems

Problem 6 (IMOSL 2016 C1). *The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then* *secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leader's in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses* 101*, the deputy leader would write down* $001, 111$ *and* 100*.) The contestant is allowed to look at the strings written by the deputy leader and guess the leader's string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?*

Problem 7 (India TST 2024)**.** *A sleeping rabbit lies in the interior of a convex* 2024*-gon. A hunter picks three vertices of the polygon and he lays a trap which covers the interior and the boundary of the triangular region determined by them. Determine the minimum number of times he needs to do this to guarantee that the rabbit will be trapped.*

Problem 8 (Canada MO 2024 P4)**.** *Treasure was buried in a single cell of an $M \times N$ ($2 \leq M$, N) grid. Detectors were brought to find the cell with the treasure. For each detector, you can set it up to scan a specific subgrid $[a, b] \times [c, d]$ with $1 \leq a \leq b \leq M$ and $1 \leq c \leq d \leq N$. Running the detector will tell you whether the treasure is in the region or not, though it cannot say where in the region the treasure was detected. You plan on setting up Q detectors, which may only be run simultaneously after all Q detectors are ready.*

In terms of M and N, what is the minimum Q required to gaurantee to determine the location of the treasure?

Problem 9 (Korea MO 2024 P3)**.** *Let S be a set consisting of* 2024 *points on a plane, such that no three points in S are collinear. A line ℓ passing through any two points in S is called a "weakly balanced line" if it satisfies the following condition:*

(Condition) When the line ℓ divides the plane into two regions, one region contains exactly 1010 *points of S, and the other region contains exactly* 1012 *points of S (where each region contains no points lying on ℓ).*

Let $\omega(S)$ denote the number of weakly balanced lines among the lines passing through pairs of points in S. Find the smallest possible value of $\omega(S)$.