Invariants and Monovariants

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1 Lecture Problems

Problem 1 (Mimamsa 2019)**.** *There are* 1000 *glasses of wine,* 500 *each of red and blue. At each step, you choose two glasses and mix them up. Regardless of quantity, if you mix a red with a red, it becomes blue. Instead, if you mix a red with a blue, then it becomes blue. Lastly, a blue with a blue gives blue. What will be the color of the last remaining glass?*

Problem 2. *The numbers* 1 *through* 2023*, inclusive, are written on a blackboard. A move consists of taking two written numbers a and b, erasing them, and writing $ab + a + b$ on the board. Continue this until only one number is left on the board. What is this number?*

Problem 3 (Canada MO 2024 P2)**.** *Jane writes down* 2024 *natural numbers around the perimeter of a circle. She wants the* 2024 *products of adjacent pairs of numbers to be exactly the set $\{1!, 2!, \ldots, 2024!\}$. Can she accomplish this?*

Problem 4 (PSS)**.** $2n$ *ambassadors are invited to a banquet. Every ambassador has at most $n-1$ enemies. Prove that the ambassadors can be seated around a round table, so that nobody sits next to an enemy.*

Problem 5. *Given a graph G on n vertices, show that we can color the vertices with the colors red and blue so that for every vertex, at least half of its neighbors are of opposite color.*

Problem 6 (Argentina OMA 2020)**.** *Let n be a positive integer. In each of k squares on an $n \times n$ board there is an infected person. Every day, everyone with at least two infected neighbors is simultaneously infected. Find the smallest possible value of k so that after a large enough number of days, all people are infected.*

Problem 7 (Canada MO 2018 P1)**.** *Consider an arrangement of tokens in the plane, not necessarily at distinct points. We are allowed to apply a sequence of moves of the following kind: select a pair of tokens at points A and B and move both of them to the midpoint of A and B.*

We say that an arrangement of n tokens is collapsible if it is possible to end up with all n tokens at the same point after a finite number of moves. For what values of n, every arrangement of n tokens is collapsible.

2 Additional Problems

Problem 8 (INMO 2018 P5)**.** *There are $n \geq 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more* *apples than both of her neighbours combined, the teacher takes away one apple from that girl and gives one apple each to her neighbours. Prove that, this process stops after a finite number of steps. (Assume that, the teacher has an abundant supply of apples.)*

Problem 9 (IMOSL 2012 C1). *Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.*

Problem 10 (IMOSL 2014 C2). *We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m .*

Problem 11 (China MO 2022 P4). *A conference is attended by $n(n \geq 3)$ scientists. Each scientist has some friends in this conference (friendship is mutual and no one is a friend of him/herself). Suppose that no matter how we partition the scientists into two nonempty groups, there always exist two scientists in the same group who are friends, and there always exist two scientists in different groups who are friends.*

A proposal is introduced on the first day of the conference. Each of the scientists' opinion on the proposal can be expressed as a non-negative integer. Everyday from the second day onwards, each scientists' opinion is changed to the integer part of the average of his/her friends' opinions from the previous day. Prove that after a period of time, all scientists have the same opinion on the proposal.

Problem 12 (IMO 2011 P2). *Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.*

Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.