Games

Amit Kumar Mallik

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1 Lecture Problem

Problem 1 (Classic)**.** *A basket has n balls. Alice and Bob play the following game. Starting with Alice, each player removes either one, two or three balls from the basket. The player who cannot make a move loses. For what values of n, Alice has a winning strategy?*

Problem 2 (Rioplatense MO 2013)**.** *Two players Alice and Bob play alternatively in a convex polygon with $n \geqslant 5$ sides. In each turn, the corresponding player has to draw a diagonal that does not cut inside the polygon previously drawn diagonals. A player loses if after his turn, one quadrilateral is formed such that its two diagonals are not drawn. A starts the game. For each positive integer n, find a winning strategy for one of the players.*

Problem 3 (Game of Chomp)**.** *There is a chocolate bar of size $m \times n$. Alice and Bob eat the chocolate alternately, starting with Alice. On their turn, they must choose a (1×1) block and eat it, together with all the blocks that are below it and to its right. The top left block is poisoned.*

(i) For $m = n$, find a strategy for one of them to survive.

(ii) For what values of (m, n), Alice has a winning strategy. Can you find such a winning strategy.

Problem 4 (USAMO 2023 P4)**.** *A positive integer a is selected, and some positive integers are written on a board. Alice and Bob play the following game. On Alice's turn, she must replace some integer n on the board with $n + a$, and on Bob's turn he must replace some even integer n on the board with $n/2$. Alice goes first and they alternate turns. If on his turn Bob has no valid moves, the game ends.*

After analyzing the integers on the board, Bob realizes that, regardless of what moves Alice makes, he will be able to force the game to end eventually. Show that, in fact, for this value of a and these integers on the board, the game is guaranteed to end regardless of Alice's or Bob's moves.

Problem 5 (JBMO P3)**.** *Alice and Bob play the following game on a 100×100 grid, taking turns, with Alice starting first. Initially the grid is empty. At their turn, they choose an integer from 1 to 100^2 that is not written yet in any of the cells and choose an empty cell, and place it in the chosen cell. When there is no empty cell left, Alice computes the sum of the numbers in each row, and her score is the maximum of these 100 numbers. Bob computes the sum of the numbers in each column, and his score is the maximum of these 100 numbers.*

Alice wins if her score is greater than Bob's score, Bob wins if his score is greater than Alice's score, otherwise no one wins.

Find if one of the players has a winning strategy, and if so which player has a winning strategy.

2 Additional problems

Problem 6 (INMO 2023 P4)**.** *Let $k \geq 1$ and $N > 1$ be two integers. On a circle are placed $2N + 1$ coins all showing heads. Calvin and Hobbes play the following game. Calvin starts and on his move can turn any coin from heads to tails. Hobbes on his move can turn at most one coin that is next to the coin that Calvin turned just now from tails to heads. Calvin wins if at any moment there are k coins showing tails after Hobbes has made his move. Determine all values of k for which Calvin wins the game.*

Problem 7 (Iran MO 2024 P2)**.** *Sahand and Gholam play on a 1403×1403 table. Initially all the unit square cells are white. For each row and column there is a key for it (totally 2806 keys). Starting with Sahand players take turn alternatively pushing a button that has not been pushed yet, until all the buttons are pushed. When Sahand pushes a button all the cells of that row or column become black, regardless of the previous colors. When Gholam pushes a button all the cells of that row or column become red, regardless of the previous colors. Finally, Gholam's score equals the number of red squares minus the number of black squares and Sahand's score equals the number of black squares minus the number of red squares. Determine the minimum number of scores Gholam can guarantee without if both players play their best moves.*

Problem 8 (Canada MO 2019 P5)**.** *A 2-player game is played on $n \geq 3$ points, where no 3 points are collinear. Each move consists of selecting 2 of the points and drawing a new line segment connecting them. The first player to draw a line segment that creates an odd cycle loses. (An odd cycle must have all its vertices among the n points from the start, so the vertices of the cycle cannot be the intersections of the lines drawn.) Find all n such that the player to move first wins.*

Problem 9 (Balkan MO Shortlist 2023 C1)**.** *Joe and Penny play a game. Initially there are 5000 stones in a pile, and the two players remove stones from the pile by making a sequence of moves. On the k-th move, any number of stones between 1 and k inclusive may be removed. Joe makes the odd-numbered moves and Penny makes the even-numbered moves. The player who removes the very last stone is the winner. Who wins if both players play perfectly?*

Problem 10 (APMO 2022 P4)**.** *Let n and k be positive integers. Cathy is playing the following game. There are n marbles and k boxes, with the marbles labelled 1 to n. Initially, all marbles are placed inside one box. Each turn, Cathy chooses a box and then moves the marbles with the smallest label, say i, to either any empty box or the box containing marble $i + 1$. Cathy wins if at any point there is a box containing only marble n. Determine all pairs of integers (n, k) such that Cathy can win this game.*