Colorings

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1 Lecture problems

Problem 1 (Classic). *Consider a* 8×8 *grid with the two diagonally opposite corners removed. Can you cover the remaining board with* 62 *squares using* 2×1 *or* 1×2 *dominoes?*

Problem 2. *You are given a* $m\times n$ *grid with an integer on each cell. In a move, you choose an integer and add the same to two adjacent cells. Determine for which initial board configurations is it possible to get zeros on each cell in finitely many moves.*

Suppose all the integers were non-negative to start with. Now, solve the same problem with the additional constraint that any integer in any intermediate board configuration is also non-negative.

Problem 3. 65 *beetles are placed on a* 9×9 *grid. Each minute, each beetle will move to an adjacent square, either horizontally or vertically, but never twice in the same orientation. So a beetle could move Left, Up, Left, Down, but could not go Left then Right. Prove that eventually two beetles will be on the same square.*

Problem 4 (INMO 2020 P6). *A stromino is a* 3×1 *rectangle. Show that a* 5×5 *board divided into twenty-five* 1×1 *squares cannot be covered by* 16 *strominos such that each stromino covers exactly three squares of the board, and every square is covered by one or two strominos. (A stromino can be placed either horizontally or vertically on the board.)*

Problem 5 (PSS). *Remove one corner of a* $(2n+1)\times(2n+1)$ *board. For what values of* n *can the rest of the board be covered with* 2×1 *and* 1×2 *dominoes such that there are exactly half of each type.*

Problem 6 (IMOSL 2017 C1). *A rectangle* $\mathcal{R}$ *with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of* $\mathcal{R}$ *are either all odd or all even.*

Problem 7 (USAJMO 2023 P3). *Consider an* n*-by-*n *board of unit squares for some odd positive integer* n*. We say that a collection* C *of identical dominoes is a maximal grid-aligned configuration on the board if* C *consists of* $(n^2-1)/2$ *dominoes where each domino covers exactly two neighboring squares and the dominoes don't overlap:* C *then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let* $k(C)$ *be the number of distinct maximal grid-aligned configurations obtainable from* C *by repeatedly sliding dominoes. Find the maximum value of* $k(C)$ *as a function of* n*.*

2 Additional Problems

Problem 8 (IMO 1993 P3). *On an infinite chessboard, a solitaire game is played as follows: at the start, we have n^2 pieces occupying a square of side n. The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed. For which n can the game end with only one piece remaining on the board?*

Problem 9 (USAMO 2023 P3). *Consider an n-by-n board of unit squares for some odd positive integer n. We say that a collection C of identical dominoes is a maximal grid-aligned configuration on the board if C consists of $(n^2-1)/2$ dominoes where each domino covers exactly two neighboring squares and the dominoes don't overlap: C then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let $k(C)$ be the number of distinct maximal grid-aligned configurations obtainable from C by repeatedly sliding dominoes. Find all possible values of $k(C)$ as a function of n.*