INMOTC 2025 (MP region)

ALGEBRA

12 January 2025

List of problems and examples

§1 Problems

Example 1.1 (Moscow MO 1946 Grades 7–8 P5)**.** Prove that after completing the multiplication and collecting the terms

$$(1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100})(1 + x + x^2 + \cdots + x^{99} + x^{100})$$

has no monomials of odd degree.

> **Summary** — What happens if x is replaced by $-x$?

Example 1.2. Let n be an even positive integer, and let $p(x)$ be a polynomial of degree n such that $p(k) = p(-k)$ for $k = 1, 2, \ldots, n$. Prove that there is a polynomial $q(x)$ such that $p(x) = q(x^2)$.

> **Walkthrough** — Note that the polynomial $p(x) - p(-x)$ has degree $< n$ because n is even. Observe that it has at least n roots.

> **Remark.** What would happen if n is not assumed to be even?

Example 1.3. Determine the remainder when $x + x^9 + x^{25} + x^{49} + x^{81} + x^{121}$ is divided by $x^3 - x$.

Example 1.4 (Moscow MO 2015 Grade 9 P6)**.** Do there exist two polynomials with integer coefficients such that each of them has a coefficient with absolute value exceeding 2015, but no coefficient of their product has absolute value exceeding 1?

> **Summary** — Try to come up with enough polynomials $g_1(x), g_2(x), g_3(x), \ldots$ and $h_1(x), h_2(x), h_3(x), \ldots$ such that each of the products $g_1 g_2 g_3 \ldots$ and $h_1 h_2 h_3 \ldots$ have at least one coefficient which is **large in absolute value**, and all the coefficients of the product $(g_1 g_2 g_3 \ldots)(h_1 h_2 h_3 \ldots)$ are at most 1 in absolute value.

§2 Factorization and roots

Example 2.1. Let $g(x)$ and $h(x)$ be polynomials with real coefficients such that

$$g(x)(x^2 - 3x + 2) = h(x)(x^2 + 3x + 2)$$

and $f(x) = g(x)h(x) + (x^4 - 5x^2 + 4)$. Prove that $f(x)$ has at least four real roots.

 The content posted here and at this blog by Evan Chen are quite useful.

Example 2.2 (USAMO 1975 P3)**.** A polynomial $P(x)$ of degree n satisfies

$$P(k) = \frac{k}{k+1} \quad \text{for } k = 0, 1, 2, \ldots, n.$$

Find $P(n+1)$.

Example 2.3. Let $P(x)$ be a polynomial with real coefficients. Assume that $P(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist polynomials g, h with real coefficients such that

$$P = g^2 + h^2.$$

Walkthrough —

(a) Show that the real roots of P have even multiplicity.

(b) Conclude that P can be expressed as a product of monic quadratic polynomials with real coefficients having nonreal roots, and even powers of degree one polynomials with real coefficients.

(c) Show that a monic quadratic polynomial with real coefficients having nonreal roots is the sum of the squares of two polynomials with real coefficients.

§3 Roots of unity

Example 3.1 (USAMO 2014 P1)**.** Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$ can take.

Example 3.2. Let $P(x)$ be a monic polynomial with integer coefficients such that all its zeroes lie on the unit circle. Show that all the zeroes of $P(x)$ are roots of unity, that is, $P(x)$ divides $(x^n - 1)^k$ for some positive integers n, k.

Example 3.3 (USAMO 1976 P5)**.** If $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ are all polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x),$$

prove that $x - 1$ is a factor of $P(x)$.

Example 3.4 (Leningrad Math Olympiad 1991)**.** A finite sequence $a_1, a_2, \ldots, a_n$ is called p-balanced if any sum of the form

$$a_k + a_{k+p} + a_{k+2p} + \ldots$$

is the same for any $k = 1, 2, 3, \ldots$. For instance the sequence

$$a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, a_5 = 3, a_6 = 2$$

is a 3-balanced. Prove that if a sequence with 50 members is p-balanced for $p = 3, 5, 7, 11, 13, 17$, then all its members are equal zero.

Summary — Consider the polynomial $\sum_{i=1}^{n} a_i x^n$.

§4 Growth of polynomials

Example 4.1 (India RMO 2015b P3)**.** Let $P(x)$ be a nonconstant polynomial whose coefficients are positive integers. If $P(n)$ divides $P(P(n) - 2015)$ for all natural numbers n, then prove that $P(-2015) = 0$.

Summary — In absolute value, a higher degree polynomial dominates a smaller degree polynomial at arguments which are large enough in absolute value.

§5 Size of the roots

Example 5.1. Find all polynomials P (with complex coefficients) satisfying

$$P(x)P(x+2) = P(x^2).$$

Summary — Note that if α is a root of P, then so are α^2 and $(\alpha-2)^2$. Also note that if $\alpha \neq 1$, then $|(\alpha-2)^2| > |\alpha|$. Conclude that $P(x) = c(x-1)^n$.

Example 5.2 (INMO 2018 P4)**.** Find all polynomials $P(x)$ with real coefficients such that $P(x^2 + x + 1)$ divides $P(x^3 - 1)$.

Walkthrough —

(a) Show that if α is a root of $P(x)$, then $P(x)$ vanishes at $(\beta_1 - 1)\alpha$ and $(\beta_2 - 1)\alpha$, where β_1, β_2 are the roots of $x^2 + x + 1 = \alpha$.

(b) If α is nonzero, then show that one of $(\beta_1 - 1)\alpha$ and $(\beta_2 - 1)\alpha$ is larger than α in absolute value.

Example 5.3. Does there exist a polynomial $f(x)$ satisfying

$$xf(x-1) = (x+1)f(x)?$$

§6 Differentiation and double roots

Lemma 1

Let $P(x)$ be a polynomial with complex coefficients, and α be a complex number. Then α is a double root of $P(x)$ (i.e., $(x-\alpha)^2$ divides $P(x)$) if and only if it is a root of $P(x)$ and $P'(x)$.

Example 6.1. Let $P(x), Q(x)$ be polynomials with complex coefficients such that they have the same set of roots with possibly different multiplicities. Suppose that $P+1, Q+1$ also have the same set of roots with possibly different multiplicities. Show that $P = Q$.

Walkthrough —

(a) Assume that $\deg P \geq \deg Q$.

(b) Denote these two set of roots by S_1, S_2. Considering multiplicities, show that

$$2 \deg P - |S_1| - |S_2| \leq \deg P' = \deg P - 1,$$

which yields

$$|S_1| + |S_2| > \deg P.$$

(c) Note that $P - Q$ vanishes at the elements of $S_1 \cup S_2$, which has size larger than the degree of $P - Q$.

§7 Crossing the x-axis/Intermediate value theorem

Example 7.1 (China TST 1995)**.** Alice and Bob play a game with a polynomial of degree at least 4:

$$x^{2n} + \square x^{2n-1} + \square x^{2n-2} + \cdots + \square x + 1.$$

They fill in real numbers to empty boxes in turn. If the resulting polynomial has no real root, Alice wins; otherwise, Bob wins. If Alice goes first, who has a winning strategy?

§8 Lagrange interpolation

Example 8.1. If a polynomial of degree n takes rationals to rationals on $n+1$ points, then show that it is a rational polynomial.

Example 8.2 (USAMO 2002 P3)**.** Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

Example 8.3. For each positive integer $n \geq 1$, determine all monic polynomials of degree n whose roots are all real, for which every coefficient is either 1 or -1.

Walkthrough —

(a) Consider the average of the squares of the roots, and show that it is small (and consequently, smaller than their geometric mean) if the polynomial has degree ≥ 4.

(b) Repeat the argument for degree three polynomials.

(c) Finding the degree one and degree two Polynomials is easy.

§9 Integer divisibility

Lemma 2

If P is a polynomial with integer coefficients and a, b are integers, then $P(a) - P(b)$ is a multiple of $a - b$.

Example 9.1 (USAMO 1974 P1). Let a, b, and c denote three distinct integers, and let P denote a polynomial having all integral coefficients. Show that it is impossible that $P(a) = b$, $P(b) = c$, and $P(c) = a$.

Example 9.2. Let $P(x)$ be a polynomial with integer coefficients, and let n be an odd positive integer. Suppose that $x_1, x_2, \ldots, x_n$ is a sequence of integers such that $x_2 = P(x_1), x_3 = P(x_2), \ldots, x_n = P(x_{n-1})$, and $x_1 = P(x_n)$. Prove that all the x_i's are equal.

Walkthrough — Show that

$$a_1 - a_2 \mid a_2 - a_3 \mid a_3 - a_4 \mid \cdots \mid a_n - a_1 \mid a_1 - a_2.$$

Note that sum of these differences is an odd multiple of their absolute value.

Lemma 3

Let P be a polynomial with integer coefficients. Suppose a is an integer and k is a positive integer such that $P(P(\ldots P(P(a))\ldots)) = a$, where P occurs k times. Show that $P(P(a)) = a$.

Example 9.3 (IMO 2006 P5). (Dan Schwarz, Romania) Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients, and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\dots P(P(x))\dots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.