INMO 2026

Exercise 1 (INMO 2026 P1,). Let $x_1, x_2, x_3, \ldots$ be a sequence of positive integers defined as follows: $x_1 = 1$ and for each $n \geq 1$, we have

$$x_{n+1} = x_n + \lfloor \sqrt{x_n} \rfloor.$$

Determine all positive integers m for which $x_n = m^2$ for some $n \geq 1$. (Here $\lfloor x \rfloor$ denotes the greatest integer less or equal to x for every real number x.)

Walkthrough —

(a)

Solution 1. Let us establish the following claim.

Claim — For any integer $m \geq 2$, the inequalities

$$(m+r)^2 < m^2 + m + 2m + 2(m+1) + \cdots + 2(m+r-1),$$
$$m^2 + m + 2m + 2(m+1) + \cdots + 2(m+r-1) + m + r < (m+r+1)^2$$

hold for any integer r with $1 \leq r \leq m-1$.

Proof of the Claim. Note that

$$\begin{aligned} 2^2 + 2 + 2 \cdot 2 &= 10 \\ &> (2+1)^2, \\ 2^2 + 2 + 2 \cdot 2 + 2 + 1 &= 13 \\ &< (2+1+1)^2 \end{aligned}$$

holds, which shows that the claim is true for $m = 2$. Assume that the claim is true for some integer $m \geq 2$. Then, for any integer r with $1 \leq r \leq m-1$, we have

$$(m+1)^2 + (m+1) + 2(m+1) + 2(m+1+1) + \cdots + 2(m+1+r-1)$$

$$
\begin{aligned}
&= m^2 + m + 2m + 2(m+1) + \cdots + 2(m+r-1) + 2m + 1 + 1 + 2r \\
&> (m+r)^2 + 2m + 2r + 2 \quad \text{(by the induction hypothesis)} \\
&> (m+r+1)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&(m+1)^2 + (m+1) + 2(m+1) + 2(m+1+1) \\
&\quad + \cdots + 2(m+1+r-1) + (m+1+r) \\
&= m^2 + m + 2m + 2(m+1) + \cdots + 2(m+r-1) + m + r \\
&\quad + 2m + 1 + 1 + 2r + 1 \\
&< (m+r+1)^2 + 2m + 2r + 3 \quad \text{(by the induction hypothesis)} \\
&< (m+r+2)^2.
\end{aligned}
$$

Moreover, the case $r = m$ can be verified directly as follows:

$$
\begin{aligned}
&(m+1)^2 + (m+1) + 2(m+1) + 2(m+1+1) \\
&\quad + \cdots + 2(m+1+m-1) \\
&= (m+1)^2 + (m+1) + (3m+1)m \\
&= m^2 + 2m + 1 + m + 1 + 3m^2 + m \\
&= 4m^2 + 4m + 2 \\
&> (m+1+m)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&(m+1)^2 + (m+1) + 2(m+1) + 2(m+1+1) \\
&\quad + \cdots + 2(m+1+m-1) + (m+1+m) \\
&= 4m^2 + 4m + 2 + (2m+1) \\
&= 4m^2 + 6m + 3 \\
&< (m+1+m+1)^2.
\end{aligned}
$$

This completes the induction step, and hence the proof of the claim. □

Consider the map $f\colon \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ defined by

$$f(k) = k + \lfloor \sqrt{k} \rfloor.$$

Let $m \geq 2$ be a positive integer. Consider the sequence defined by $y_1 = m^2$ and $y_{n+1} = f(y_n)$ for each $n \geq 1$. Note that

$$
\begin{aligned}
y_2 &= m^2 + m, \\
y_3 &= m^2 + m + m
\end{aligned}
$$

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holds. Using $m^2 < y_3 < (m+1)^2$, and applying the Claim, it follows for any integer r with $1 \le r \le m-1$ that

$$y_{3+2r-1} = m^2 + m + 2m + 2(m+1) + \cdots + 2(m+r-1),$$

and

$$y_{3+2r} = m^2 + m + 2m + 2(m+1) + \cdots + 2(m+r-1) + m + r.$$

Since $m \ge 2$, applying the Claim, we see that

$$y_4, y_5, \ldots, y_{2m+1}$$

are not perfect squares. Note that y_2, y_3 are not perfect squares. This shows that none of

$$y_2, y_3, \ldots, y_{2m+1}$$

is a perfect square. Moreover, we have

$$\begin{aligned} y_{2m+2} &= y_{2m+1} + (m + m - 1) \\ &= m^2 + m + 2m + 2(m+1) + \cdots + 2(m+m-2) + m + m - 1 \\ &\quad + (m + m - 1) \\ &= m^2 + m + (3m-2)(m-1) + 4m - 2 \\ &= (2m)^2, \end{aligned}$$

which is a perfect square. Therefore, starting from any perfect square m^2 with $m \ge 2$, we reach the next perfect square $(2m)^2$ through repeated application of the map f, without encountering any other perfect square in between.

Note that

$$x_1 = 1^2, x_2 = 2, x_3 = 2 + 1 = 3, x_4 = 3 + 1 = 4 = 2^2.$$

Thus, the terms of the sequence (x_n) that are perfect squares are precisely the squares of the non-negative powers of 2. In other words, the positive integers m for which $x_n = m^2$ for some $n \ge 1$ are precisely the non-negative powers of 2. ■

Exercise 2 (INMO 2026 P2,). Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying the following condition: for each $k > 2026$, the number $f(k)$ equals the maximum number of times a number appears in the list $f(1), f(2), \ldots, f(k-1)$. Prove that $f(n) = f(n + f(n))$ for infinitely many $n \in \mathbb{N}$. (Here $\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$ of positive integers.)

Walkthrough —

(a) Show that $f|_{\{2027,2028,\ldots\}}$ is a non-decreasing function.

(b) Show that f takes arbitrarily large values.

(c) Show that there exists an integer $r \geq 2027$ such that

$$f(r) > \max\{f(1), f(2), \ldots, f(r-1)\}.$$

(d) Show that for such an integer r, the values of f at the integers $r, r+1, r+2, \ldots, r+f(r)$ are all equal, and $f(r+f(r)+1) = f(r)+1$.

(e) Conclude the solution.

Solution 2. Note that $f(n) \leq f(n+1)$ holds for any $n \geq 2027$, which shows that $f|_{\{2027,2028,\ldots\}}$ is a non-decreasing function. For any $m \geq 2027$, note that $f(m)$ is a positive integer. Observe that for any $m \geq 2027$, if

$$f(m), f(m+1), f(m+2), \ldots, f(m+f(m))$$

are all equal, then $f(m+f(m)+1)$ is greater than or equal to $f(m)+1$. This shows that

$$f(m) \geq 1, \quad f(m) < f(m+f(m)+1)$$

hold for any $m \geq 2027$. This shows that f takes arbitrarily large values.

Claim — Let $r \geq 2027$ be an integer such that

$$f(r) > \max\{f(1), f(2), \ldots, f(r-1)\}.$$

Then f takes equal values at the integers

$$r, r+1, r+2, \ldots, r+f(r)$$

and $f(r+f(r)+1) = f(r)+1$.

Proof of the Claim. The maximum number of times that a number appears in the list $f(1), f(2), \ldots, f(r-1)$ is equal to $f(r)$. Using the inequality $f(r) > \max\{f(1), f(2), \ldots, f(r-1)\}$, we see that any such number is less than $f(r)$. Therefore, the numbers $f(r), f(r+1), \ldots, f(r+f(r))$ are equal. It follows that $f(r+f(r)+1) = f(r)+1$. □

Since f takes arbitrarily large values, there exists an integer $r \geq 2027$ such that $f(r) > \max\{f(1), f(2), \ldots, f(r-1)\}$. Using the Claim and applying induction, it follows that the images of $r, r+1, r+2, \ldots$ under f are equal to

$$\underbrace{t, t, \ldots, t}_{t+1 \text{ times}}, \underbrace{t+1, t+1, \ldots, t+1}_{t+2 \text{ times}}, \underbrace{t+2, t+2, \ldots, t+2}_{t+3 \text{ times}}, \ldots$$

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respectively, where $t = f(r)$. This implies that $f(n) = f(n+f(n))$ for infinitely many $n \in \mathbb{N}$, as desired. ■

Exercise 3 (INMO 2026 P4,). Two integers a and b are called *companions* if every prime number p either divides both or none of a, b. Determine all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(0) = 0$ and the numbers $f(m) + n$ and $f(n) + m$ are companions for all $m, n \in \mathbb{N}_0$. (Here $\mathbb{N}_0$ denotes the set of all non-negative integers.)

It is worth having a look at IMOSL 2007 N5.

Walkthrough —

(a)

Solution 3. Let m, n be elements of $\mathbb{N}_0$. Let p be a prime number dividing $f(m)-f(n)$. Let k be a positive integer such that p divides the integers $f(m)+k$ and $f(n)+k$. Then p also divides the integers $m+f(k)$ and $n+f(k)$, and hence p divides $m-n$. Thus, for any prime number p, if p divides $f(m)-f(n)$, then p divides $m-n$.

In particular, for any integer $n \geq 0$, it follows that $f(n+1)-f(n)$ is equal to one of $1, -1$. We claim that $f(n+1)-f(n) = 1$ for all $n \geq 0$. On the contrary, suppose that there exists an integer $n \geq 0$ such that $f(n+1)-f(n) = -1$. Let p be a prime number greater than or equal to $n+1$. This gives

$$\begin{aligned} f(p) &= \sum_{i=1}^{p} (f(i) - f(i-1)) \\ &\leq p - 2. \end{aligned}$$

Since $f(p) + 0, p + f(0)$ are companions, it follows that $f(p)$ is a positive multiple of p, which contradicts the inequality $f(p) \leq p-2$. Thus, we have $f(n+1) - f(n) = 1$ for all $n \geq 0$, and hence $f(n) = n$ for all $n \geq 0$. It is easy to verify that this function satisfies the required conditions. Therefore, the only function satisfying the required conditions is

$$f(n) = n \quad \text{for all } n \in \mathbb{N}_0.$$

■