INMO 2025

Exercise 1 (INMO 2025 P5, , proposed by Pranjal Srivastava and Rohan Goyal)**.** Greedy goblin Griphook has a regular 2000-gon, whose every vertex has a single coin. In a move, he chooses a vertex, removes one coin each from the two adjacent vertices, and adds one coin to the chosen vertex, keeping the remaining coin for himself. He can only make such a move if both adjacent vertices have at least one coin. Griphook stops only when he cannot make any more moves. What is the maximum and minimum number of coins that he could have collected?

Walkthrough —

(a) Show that 1998 coins can be collected (see Fig. 1).

(b) Show that the difference of the number of coins at the vertices with odd indices, and the number of coins at the vertices with even indices, does not change modulo 3 after any move.

(c) Conclude that no more than 1998 coins can be collected.

(d) Show that 668 coins can be collected.

(e) Show that at least 668 coins are collected when Griphook stops.

The following is from AoPS, and is due to Rohan Goyal.

Solution 1. Denote the regular 2000-gon by $A_1A_2 \dots A_{2000}$. The claims below imply that the maximum (resp. minimum) number of coins that Griphook could have collected is 1998 (resp. 668).

Claim — No more than 1998 coins can be collected.

Proof of the Claim. Note that after any move, the number of coins at the vertices with odd indices and the number of coins at the vertices with even indices either reduces by 2 and increases by 1 respectively, or increases by 1 and reduces by 2 respectively. This shows that the difference of the number of coins at the vertices with odd indices and the number of coins at the vertices with even indices, does not change modulo 3 after any move, and hence, this difference is zero modulo 3 after any move.

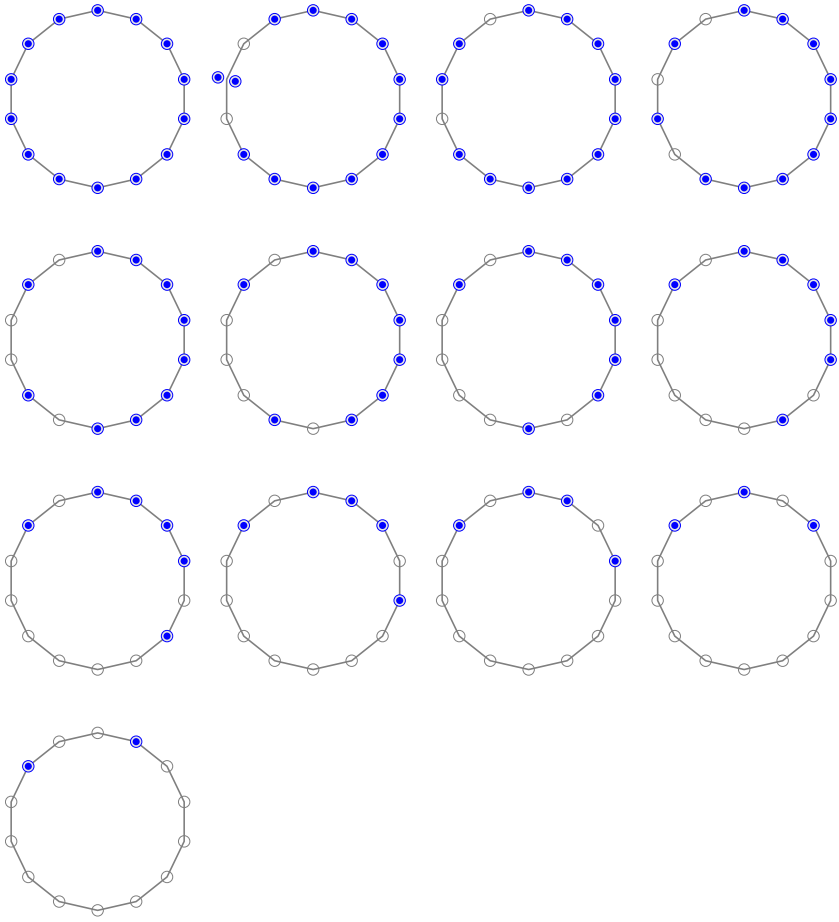


Figure 1: INMO 2025 P5, Collecting 12 coins, Exercise 1

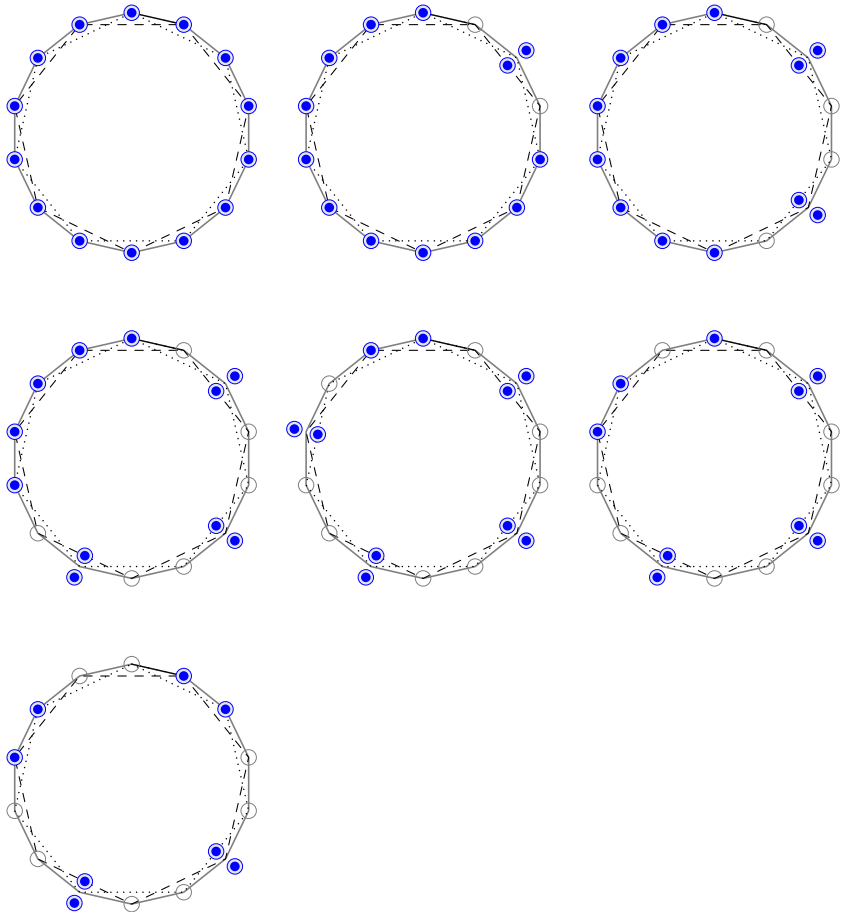


Figure 2: INMO 2025 P5, Collecting 6 coins, Exercise 1

Since the number of coins remaining after any move is at least 1, there are at least two coins left at the end of any move. Consequently, no more that 1998 coins can be collected. □

Claim — There is a way to collect 1998 coins.

Proof of the Claim. Let us denote the number of coins at the vertices of the regular 2000-gon by a string of length 2000, consisting of nonnegative integers, where the i-th entry of the string denotes the number of coins at the i-th vertex A_i.

In the following, 1^m (resp. 0^m) denotes a string of m 1's (resp. 0's). Note that by making a move, the string 1^{2000} can be turned to $1^{1996}0201$, which can be turned to $1^{1995}01010$ by making another move. Observe that the sub-string 101 can be turned to 010 by making a move. Hence, making enough suitable moves, the string $1^{1995}01010$ can be reduced to $010^{1996}10$. While reaching $010^{1996}10$ from 1^{2000}, a total of 1998 coins have been earned, proving the claim. □

Claim — There is a sequence of moves, which leads to the collection of 668 coins when Griphook stops.

Proof of the Claim. Write the string 1^{2000} as $1(111)^{666}1$, which can be turned to $1(020)^{666}1$, which is equal to $1020(020)^{664}0201$. This can be reduced to $0110(020)^{664}0110$. Note that no further move is possible, and hence Griphook has to stop, and he earned $2000 - 4 - 2 \times 664 = 668$ coins. □

Claim — For any sequence of moves, Griphook will have at least 668 coins after he stops.

Proof of the Claim. Let c_i denote the number of coins at the i-th vertex. Note that each move reduces the sum

$$S = \sum_{i=1}^{2000} \min\{c_{i-1}, c_{i+1}\}.$$

In this sum, the indices are to be read modulo 2000. In each move, this sum reduces at least by 1, and Griphook stops precisely when this sum reduces to 0. Note that at any moment, for any two one-vertex apart vertices, one of them contains at most one coin[1], that is,

$$\min\{c_{i-1}, c_{i+1}\} \leq 1$$

[1] Why?

The content posted here and at this blog by Evan Chen are quite useful.

for any $1 \leq i \leq 2000$. Write

$$E = \sum_{1 \leq i \leq 2000, 2 | i} \min \{c_{i-1}, c_{i+1}\},$$

$$O = \sum_{1 \leq i \leq 2000, 2 \nmid i} \min \{c_{i-1}, c_{i+1}\}.$$

Note that in each move, one of E, O is reduced at most by 3. Hence, at least 334 moves are required to reduce any of E, O to 0. Since in each move, only one of E, O gets reduced, it follows that Griphook makes at least $334 + 334 = 668$ moves before he stops.

□

■