INMO 2018

Exercise 1 (INMO 2018 P4,). Find all polynomials $P(x)$ with real coefficients such that $P(x^2+x+1)$ divides $P(x^3-1)$.

Walkthrough —

(a) Show that if α is a root of $P(x)$, then $P(x)$ vanishes at $(\beta_1-1)\alpha$ and $(\beta_2-1)\alpha$, where β_1, β_2 are the roots of $x^2+x+1=\alpha$.

(b) If α is nonzero, then show that one of $(\beta_1-1)\alpha$ and $(\beta_2-1)\alpha$ is larger than α in absolute value.

Solution 1. Let us establish the following claim.

Claim — Let α denote a nonzero root of $P(x)$ in $\mathbb{C}$. Then for some $\beta \in \mathbb{C}$ satisfying $\beta^2+\beta+1=\alpha$, the element $(\beta-1)\alpha$ is a root of $P(x)$ and is larger than α in absolute value.

Proof of the Claim. Let β_1, β_2 denote the roots of $x^2+x+1=\alpha$ in $\mathbb{C}$. Since $P(x^2+x+1)$ divides the polynomial $P(x^3-1)$, it follows that for any $1 \le i \le 2$,

$$\beta_i^3 - 1 = (\beta_i - 1)(\beta_i^2 + \beta_i + 1) = (\beta_i - 1)\alpha$$

is a root of $P(x)$. Noting that

$$\begin{aligned} |\beta_1 - 1| + |\beta_2 - 1| &\ge |\beta_1 + \beta_2 - 2| \\ &= |-1-2| \\ &> 2, \end{aligned}$$

we obtain that at least one of $\beta_1 - 1, \beta_2 - 1$ has absolute value larger than 1. Consequently, $|(\beta_i-1)\alpha| > |\alpha|$ holds for some $1 \le i \le 2$. This proves the Claim. □

If $P(x)$ has a nonzero root in $\mathbb{C}$, then applying the above Claim to a root of $P(x)$ of largest absolute value, we would obtain a contradiction. Hence, $P(x)$ is equal to cx^n for some $c \in \mathbb{R}$ and an integer $n \ge 0$. Moreover, any polynomial of this form satisfies the given condition. Consequently, these are all the polynomials satisfying the given condition. ■

Remark. Note that proving $|(\beta_1 - 1)(\beta_2 - 1)| > 1$, in order to conclude that at least one of $\beta_1 - 1, \beta_2 - 1$ has absolute value larger than 1, does not seem to work. Moreover, $|(\beta_1 - 1)(\beta_2 - 1)|$ is smaller than $\frac{1}{2}(|\beta_1 - 1| + |\beta_2 - 1|)$. **Unsurprisingly**, a lower bound for the bigger quantity can easily be obtained.

The content posted here and at this blog by Evan Chen are quite useful.