INMO 2015

Exercise 1 (INMO 2015 P6,). From a set of 11 square integers, show that one can choose 6 numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that

$$a^2 + b^2 + c^2 \equiv e^2 + f^2 + g^2 \pmod{12}.$$

> **Walkthrough** —
>
> **(a)**

Solution 1. Let A denote the set of the given 11 square integers. Let n denote the largest positive integer such that some n elements of A are congruent to each other modulo 12. Let X_n denote the set formed by some such n integers. Let x be an element of $\{0, 1, 4, 9\}$ which is congruent to any element of X_n modulo 12. If $n = 6$, then sum of any three of those 6 integers is congruent to the remaining three integers modulo 12. Note that any perfect square is congruent to one of $0, 1, 4, 9$ modulo 12.

Now let us consider the case that $n = 4$ or $n = 5$. Note that any element of $A \setminus X_n$ is congruent to some element of $\{0, 1, 4, 9\} \setminus \{x\}$ modulo 12. Since $A \setminus X_n$ has $11 - n \geq 6$ elements, it follows by the pigeonhole principle that some two elements x, y of $A \setminus X_n$ are congruent modulo 12. Note that the sum of x and any two elements of X_n is congruent to the sum of y and any two of the remaining elements of X_n.

It remains to consider the case that $n \leq 3$, which we assume from now on. Note that $n \geq 3$, since the set A has 11 elements and there are only four quadratic residues modulo 12. This shows that $n = 3$. Hence, some two elements of A are congruent to an element s of $\{0, 1, 4, 9\}$ modulo 12, and for any element t of $\{0, 1, 4, 9\} \setminus \{s\}$, some three elements a_t, b_t, c_t of A are congruent to t modulo 12. It follows that

$$\sum_{t \in \{0,1,4,9\} \setminus \{s\}} a_t \equiv \sum_{t \in \{0,1,4,9\} \setminus \{s\}} b_t \pmod{12}.$$

This completes the proof. ∎