# INMO 2013

**Exercise 1** (INMO 2013 P4, , Putnam 2002 A3)**.** [**AF13**, Exercise 4.8, p. 90] Let $n$ be an integer greater than 1 and let $T_n$ be the number of nonempty subsets $S$ of $\{1, 2, \ldots, n\}$ with the property that the average of the elements of $S$ is an integer. Prove that $T_n - n$ is always even.

**Walkthrough** —

**(a)** Show that the average of any nonempty subset of $\{1, 2, \ldots, n\}$ lies between 1 and $n$.

**(b)** Observe that it suffices to show that for any $1 \leq a \leq n$, the subsets of $\{1, 2, \ldots, n\}$ with average equal to $a$ is an odd number.

**(c)** Show that for any $1 \leq a \leq n$, the nonempty subsets of $\{1, 2, \ldots, n\}$ having average equal to $a$ and not containing $a$, are in one-to-one correspondence with the subsets of $\{1, 2, \ldots, n\}$ having average equal to $a$, and which contain $a$. (Consider the averages of $\{1, 2, 3\}$ and $\{2\}$.)

**Solution 1.** Let $\mathcal{T}_n$ denote the set of nonempty subsets of $\{1, 2, \ldots, n\}$ having integer average. Note that the average of the elements of any subset of $\{1, 2, \ldots, n\}$ lies between 1 and $n$. So if a nonempty subset $S$ of $\{1, 2, \ldots, n\}$ has integer average $a$ and $S$ does not contain $a$, then $S \cup \{a\}$ is also a nonempty subset of $\{1, 2, \ldots, n\}$ with average $a$. Hence, for any integer $1 \leq a \leq n$, there is a bijection between the nonempty subsets of $X$ with integer average $a$ that does not contain $a$ and the subsets of $X$ with integer average $a$ that contain $a$, but not equal to $\{a\}$. So the subsets of $\mathcal{T}_n$ with average equal to $a$ is an odd number for any $1 \leq a \leq n$. Hence $T_n - n$ is even. ■

## References

[**AF13**] T. Andreescu and Z. Feng. *A Path to Combinatorics for Undergraduates: Counting Strategies*. Birkhäuser Boston, 2013. ISBN: 9780817681548. URL: https://books.google.de/books?id=3mwQBwAAQBAJ (cited p. 1)