INMO 2002

Exercise 1 (INMO 2002 P2,). Find the smallest positive value taken by $a^3+b^3+c^3-3abc$ for positive integers a, b, c. Find all a, b, c which give the smallest value.

Walkthrough —

(a) Note that $a=b=c=1$ won't work, not even taking all of a, b, c to be equal would be of any use. In other words, at least two of a, b, c have to be unequal.

(b) By taking $a=1, b=2, c=1$, one can find that $a^3+b^3+c^3-3abc=4$. Next, we need determine whether $a^3+b^3+c^3-3abc$ can be equal to $1, 2, 3$ or 4 for positive integers a, b, c.

(c) Use

$$\begin{aligned} a^3+b^3+c^3-3abc &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \\ &= \frac{1}{2}(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2) \end{aligned}$$

to get a lower bound on $a^3+b^3+c^3-3abc$.

Solution 1. Let a, b, c be positive integers such that $a^3+b^3+c^3-3abc$ is positive. Note that they cannot be equal, and hence at least two of them are distinct. Since $a^3+b^3+c^3-3abc$ is symmetric[1] in a, b, c, we may assume[2] that $a \neq b$.

Apart from the integers a and b, there is another pair of two integers among a, b, c which are not equal, that is, $b \neq c$ or $c \neq a$ holds. Indeed, if both of these two inequalities fail to hold, then $b=c$ and $c=a$ hold, and then we would have $a=b$, which is a contradiction. Note that

$$\begin{aligned} &a^3+b^3+c^3-3abc \\ &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \end{aligned}$$

[1] A reader unfamiliar with this term may require to look online.
[2] How we may do so? It does require a thought.

$$= \frac{1}{2}(a+b+c)\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$
$$\geq \frac{1}{2}(a+b+c)(1^2+1^2)$$
(since at least two of $a-b, b-c, c-a$ are nonzero, and $a+b+c>0$)
$$\geq a+b+c$$
$$\geq 1+2+1 \quad \text{(since at least two of } a-b, b-c, c-a \text{ are nonzero, and } a,b,c \geq 1)$$
$$= 4.$$

Also note that if $c > 1$, then

$$a^3+b^3+c^3-3abc > 4.$$

For $a=1, b=2, c=1$, we obtain

$$a^3+b^3+c^3-3abc = 4.$$

Hence, the smallest positive value taken by $a^3+b^3+c^3-3abc$, for positive integers a, b, c, is equal to 4.

Moreover, if a, b, c are positive integers such that $a^3+b^3+c^3-3abc$ takes the value 4, then at least two of a, b, c are unequal, and the above argument shows that

$$a+b+c \leq a^3+b^3+c^3-3abc \leq 4,$$

and consequently, two of a, b, c are equal to 1 and the remaining one is equal to 2. Hence, $a^3+b^3+c^3-3abc$ takes the value 4 precisely when

$$(a,b,c) = (1,1,2), (1,2,1), (2,1,1).$$

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The content posted here and at this blog by Evan Chen are quite useful.