INMO 1996

Exercise 1 (INMO 1996 P3,). Find all possible real numbers a, b, c, d, e which satisfy the following set of equations:

$$3a = (b+c+d)^3 \tag{1}$$
$$3b = (c+d+e)^3 \tag{2}$$
$$3c = (d+e+a)^3 \tag{3}$$
$$3d = (e+a+b)^3 \tag{4}$$
$$3e = (a+b+c)^3. \tag{5}$$

Solution 1. Let a, b, c, d, e be real numbers satisfying the given equations. Note that subtracting Eq. (2) from Eq. (1) yields

$$\begin{aligned}
3(a-b) &= (b-e)((b+c+d)^2 + (b+c+d)(c+d+e) + (c+d+e)^2) \\
&= (b-e)(3(c+d)^2 + (c+d)(2b+b+e+2e) + (b^2+be+e^2)) \\
&= (b-e)(3(c+d)^2 + 3(c+d)(b+e) + (b^2+be+e^2)) \\
&= 3(b-e)((c+d)^2 + (c+d)(b+e) + (b^2+be+e^2)/3) \\
&= 3(b-e)((c+d+(b+e)/2)^2 - ((b+e)/2)^2 + (b^2+be+e^2)/3) \\
&= 3(b-e)((c+d+(b+e)/2)^2 + \frac{1}{12}(b^2-2be+e^2)),
\end{aligned}$$

that is,

$$a-b = (b-e)\left(\left(c+d+\frac{b+e}{2}\right)^2 + \frac{1}{12}(b-e)^2\right). \tag{6}$$

Similarly, subtracting Eq. (3) from Eq. (2), Eq. (4) from Eq. (3), Eq. (5) from Eq. (4), Eq. (1) from Eq. (5) give

$$b-c = (c-a)\left(\left(d+e+\frac{c+a}{2}\right)^2 + \frac{1}{12}(c-a)^2\right), \tag{7}$$

$$c-d = (d-b)\left(\left(e+a+\frac{d+b}{2}\right)^2 + \frac{1}{12}(d-b)^2\right), \tag{8}$$

$$d-e=(e-c)\left(\left(a+b+\frac{e+c}{2}\right)^2+\frac{1}{12}(e-c)^2\right), \tag{9}$$

$$e-a=(a-d)\left(\left(b+c+\frac{a+d}{2}\right)^2+\frac{1}{12}(a-d)^2\right). \tag{10}$$

Let us consider the case that $a=b$. Then $b=e$ holds by Eq. (6), and hence, we get $a=b=e$. This gives $a=d$ by Eq. (10), and we obtain $a=b=d=e$. Then $e=c$ holds by Eq. (9). This yields $a=b=c=d=e$. Using Eq. (1), we get $3a=(3a)^3$, which implies that $a=0,\pm\frac{1}{3}$. So the solution (a,b,c,d,e) is equal to one of

$$(0,0,0,0,0),\left(\pm\frac{1}{3},\pm\frac{1}{3},\pm\frac{1}{3},\pm\frac{1}{3},\pm\frac{1}{3}\right),$$

where the signs correspond.

Now, let us consider the case that $a<b$. Since the squares are nonnegative, using Eq. (6), we get $a<b<e$. Using $a<e$ and Eq. (10), we get $d<a<b<e$. Applying Eq. (9) yields $d<a<b<e<c$. Combining Eq. (8) and $d<c$, we obtain $b<d<a<b<e<c$, which is impossible.

Note that if $a>b$ holds, then reversing the above inequalities, we would again obtain a contradiction.

The above argument proves that $a=b$, and hence, any solution (a,b,c,d,e) is equal to one of

$$(0,0,0,0,0),\left(\pm\frac{1}{3},\pm\frac{1}{3},\pm\frac{1}{3},\pm\frac{1}{3},\pm\frac{1}{3}\right).$$

Also note that any of the above is also a solution to the given equations. This proves that the above three tuples are precisely all the solutions. ■

The content posted here and at this blog by Evan Chen are quite useful.