INMO 1991

Exercise 1 (INMO 1991 P10, cf. Moscow Mathematical Olympiad 1973 Grade 8 First Round P4, UK BMO 2005 Round 2 P1)**.** For any positive integer n, let $S(n)$ denote the number of ordered pairs (x, y) of positive integers for which

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

(for instance, $S(2) = 3$). Determine the set of positive integers n for which $S(n) = 5$.

> **Walkthrough** — Is the given equation equivalent to
>
> $$(x - n)(y - n) = n^2?$$

Solution 1. For positive integers x, y, note that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

holds if and only if

$$(x - n)(y - n) = n^2$$

holds. Observe that if x, y are positive integers satisfying the given equation, then $x > n$ and $y > n$ holds. This shows that the solutions of the given equation over the positive integers are in one-to-one correspondence with the pairs of positive integers (a, b) such that $ab = n^2$, through the map

$$(a + n, b + n) \leftrightarrow (a, b).$$

Hence, the set of positive integers n satisfying $S(n) = 5$ is equal to the set of positive integers n such that n^2 has precisely 5 positive divisors. Note that any such integer n is larger than 1. Writing n as a product of powers of distinct primes, it follows that n^2 has precisely 5 positive divisors if and only if n is the square of a prime. Indeed, if $p_1, \ldots, p_r$ are distinct primes, and $a_1, \ldots, a_r$ are

positive integers, then the integer $(p_1^{a_1} \dots p_r^{a_r})^2$ has precisely 5 positive divisors if and only if

$$(2a_1 + 1)(2a_2 + 1) \dots (2a_r + 1) = 5$$

holds, which is equivalent to $r = 1, a_1 = 2$. This proves that the positive integers satisfying $S(n) = 5$ are precisely the squares of the primes. ■

The content posted here and at this blog by Evan Chen are quite useful.