INMO 1988

Exercise 1 (INMO 1988 P4,). If $a, b > 0$ with $a + b = 1$, then show that $(a + \frac{1}{a})^2 + (b + \frac{1}{b})^2 \geq \frac{25}{2}$.

Solution 1. Note that

$$\begin{aligned}\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 &= a^2 + b^2 + \frac{1}{a^2} + \frac{1}{b^2} + 4 \\ &= a^2 + b^2 + 1 + \frac{b^2}{a^2} + 2\frac{b}{a} + 1 + \frac{a^2}{b^2} + 2\frac{a}{b} + 4 \\ &= a^2 + b^2 + \left(\frac{a}{b} - \frac{b}{a}\right)^2 + \left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + 12 \\ &\geq \frac{(a+b)^2}{2} + 12 \\ &= \frac{25}{2}.\end{aligned}$$

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