INMO 1987

Exercise 1 (INMO 1987 P2,). Determine the largest number in the infinite sequence

$$1, 2^{1/2}, 3^{1/3}, 4^{1/4}, \ldots, n^{1/n}, \ldots.$$

Solution 1. Note that $1 < 2^{1/2} < 3^{1/3}$ holds. Let us establish the following Claim.

> **Claim —** For any integer $n \geq 3$, the inequality
>
> $$n^{\frac{1}{n}} > (n+1)^{\frac{1}{n+1}}$$
>
> holds.

Proof of the Claim. Observe that the inequality is equivalent to $n > \left(1 + \frac{1}{n}\right)^n$. For any integer $n \geq 3$, note that

$$\begin{aligned}
&\left(1 + \frac{1}{n}\right)^n \\
&= 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{n}\frac{1}{n^n} \\
&= 1 + \frac{n}{1!}\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} \\
&\quad + \cdots + \frac{n(n-1)(n-2)\cdots 2}{(n-1)!}\frac{1}{n^{n-1}} + \frac{n(n-1)(n-2)\ldots 1}{n!}\frac{1}{n^n} \\
&= 1 + \frac{1}{1!} + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \\
&\quad \cdots + \frac{1}{(n-1)!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-2}{n}\right) \\
&\qquad + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-2}{n}\right)\left(1 - \frac{n-1}{n}\right) \\
&< 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}
\end{aligned}$$

$$
\begin{aligned}
&< 2 + \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2} + \cdots + \frac{1}{\underbrace{2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2}_{(n-1)\text{-times}}} \\
&= 2 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} \\
&< 3 \\
&\leq n
\end{aligned}
$$

holds. This proves the Claim. □

It follows that the largest term in the given sequence is equal to $3^{1/3}$. ■

Remark. For any $n \geq e$, note that

$$n^{1/n} \geq e^{1/n} > 1 + \frac{1}{n}$$

holds, which shows that $n^{1/n} > (n+1)^{1/(n+1)}$.