IMOTC 2026

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§1 Invariants

§1.1 Impossibility

Exercise 1 (ELMO 1999 P2,). Mr. Fat moves around on the lattice points according to the following rules: From point (x, y) he may move to any of the points (y, x), $(3x, -2y)$, $(-2x, 3y)$, $(x+1, y+4)$ and $(x-1, y-4)$. Show that if he starts at $(0, 1)$ he can never get to $(0, 0)$.

> **Walkthrough** — Show that under the given moves, the modulo 5 congruence class of the sum of the coordinates of the point where Mr. Fat is located remains invariant upto a factor of 3.

Exercise 2 (RMO 2014b P6,). Let n be an odd positive integer and suppose that each square of an $n \times n$ grid is arbitrarily filled with either by 1 or by -1. Let r_j and c_k denote the product of all numbers in j-th row and k-th column respectively, $1 \le j, k \le n$. Prove that

$$\sum_{j=1}^{n} r_j + \sum_{k=1}^{n} c_k \neq 0.$$

> **Walkthrough** —
>
> **(a)** What would happen to the sum if the sign of one of the entries is changed? Does the sum change modulo 4?
>
> **(b)** What would happen when all the entries are changed to 1 by changing the signs of the negative entries?

Exercise 3 (Cono Sur Olympiad 2013 P1,). Four distinct points are marked in a line. For each point, the sum of the distances from said point to the other three is calculated; getting in total 4 numbers. Decide whether these 4 numbers can be, in some order:

(i) $29, 29, 35, 37$,

(ii) $28, 29, 35, 37$,

(iii) $28, 34, 34, 37$.

Walkthrough —

(a) For each of the four points, express the sum of the distances to the other three points in terms of the distances between consecutive points among the four points.

(b) Show that at least two of the four numbers must be equal.

(c) How does it follow that the first set of numbers is the only one that can be obtained?

Exercise 4 (Japan Mathematical Olympiad 2018 P1,). Positive integers between 1 to 100 inclusive are written on a blackboard, each exactly once. One operation involves choosing two numbers a and b on the blackboard and erasing them, then writing the greatest common divisor of $a^2 + b^2 + 2$ and $a^2b^2 + 3$. After a number of operations, there is only one positive integer left on the blackboard. Prove this number cannot be a perfect square.

Walkthrough —

(a) Show that the parity of the number of multiples of 3 on the blackboard is an invariant.

(b) Show that the last number on the blackboard is a multiple of 3, but is not a multiple of 9.

The following problem is similar to Exercise 8.

Exercise 5 (Singapore Junior Mathematical Olympiad 2019 Round 2 P2,). There are 315 marbles divided into three piles of $81, 115$ and 119. In each move Ah Meng can either merge several piles into a single pile or divide a pile with an even number of marbles into 2 equal piles. Can Ah Meng divide the marbles into 315 piles, each with a single marble?

Walkthrough —

(a)

Exercise 6 (Belarus National Olympiad 2023 Grade 8 Day 1 P1,). A move on an unordered triple of numbers (a, b, c) changes the triple to either $(a, b, 2a + 2b - c)$, $(a, 2a + 2c - b, c)$ or $(2b + 2c - a, b, c)$. Can you perform a finite sequence of moves on the triple $(3, 5, 14)$ to get the triple $(3, 13, 6)$?

Walkthrough — Show that if a sequence of moves is performed on a triple (a, b, c), then the resulting **unordered** triple is congruent to one of the triples

$$(a, b, -(a+b+c)), (a, -(a+b+c), c), (-(a+b+c), b, c), (a, b, c)$$

modulo 3, that is, the residues of the entries of the resulting triple modulo 3 coincide with the residues of the entries of one of the above triples in some order.

Remark. Note that $a^2 + b^2 + c^2 - ab - bc - ca$ is an invariant.

Exercise 7 (Belarus National Olympiad 2023 Grade 9 Day 1 P2,). A move on an unordered triple of numbers (a, b, c) changes the triple to either $(a, b, 2a + 2b - c)$, $(a, 2a + 2c - b, c)$ or $(2b + 2c - a, b, c)$. Can you perform a finite sequence of moves on the triple $(3, 5, 14)$ to get the triple $(9, 8, 11)$?

Walkthrough — Show that the mod 4 congruence class of the sum of the entries of such a triple remains invariant under the allowed moves if the sum of the entries of the initial triple is congruent to 2 modulo 4.

Remark. Note that $a^2 + b^2 + c^2 - ab - bc - ca$ is an invariant.

Exercise 8 (Kosovo National Olympiad 2023 Grade 11 P1,). In three different piles, there are 51, 49 and 5 stones, respectively. You can combine two piles in a larger pile or you can separate a pile that contains an even amount of stones into two equal piles. Is it possible that after some turns, we can separate the stones into 105 piles with 1 stone on each pile?

Walkthrough —

(a) Show that the greatest common divisor of the numbers of stones of the piles is an integer larger than 1.

(b) Show that it is impossible to separate the stones into 105 piles with 1 stone on each pile.

Exercise 9 (Canadian Mathematical Olympiad 2024 P2,). Jane writes down 2024 natural numbers around the perimeter of a circle. She wants the 2024 products of adjacent pairs of numbers to be exactly the set $\{1!, 2!, \ldots, 2024!\}$. Can she accomplish this?

Walkthrough —

(a) Show that to have such a configuration, the product $1! \cdot 2! \cdot \ldots \cdot 2024!$ must be a perfect square.

(b) Show that $1! \cdot 2! \cdot \ldots \cdot 2024!$ is not a perfect square.

Remark. Note that no prime larger than 1009 helps to show that $1! \cdot 2! \cdot \ldots \cdot 2024!$ is not a perfect square.

§1.2 Determining certain quantities

Example 10. Suppose the numbers $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{100}$ are written on a blackboard. In each step, you may erase two numbers a and b and replace them with the number $a + b + ab$. After 99 steps, only one number remains. What are the possible values of this number?

Walkthrough —

(a)

Exercise 11 (Argentina National Olympiad 2018 Level 2 P1,). A list of 2018 numbers is created using the following procedure: the first number is 47, the second number is 74, and from there, each number is equal to the number formed by the last two digits of the sum of the two previous numbers:

$$47, 74, 21, 95, 16, 11, \ldots$$

Bruno squares each of the 2018 numbers and sums them. Determine the remainder when this sum is divided by 8.

Walkthrough —

(a) Consider the residues of the numbers in the list modulo 4.

(b) Determine the residues of the squares of the numbers in the list modulo 8.

(c) Determine the remainder when the sum of the squares of the numbers in the list is divided by 8.

Exercise 12 (Balkan Mathematical Olympiad 2018 P2, , proposed by Jeremy King, UK). Let q be a positive rational number. Two ants are initially at the same point X in the plane. In the n-th minute $(n = 1, 2, \ldots)$ each of them chooses whether to walk due north, east, south or west and then walks the distance of q^n metres. After a whole number of minutes, they are at the same point in the plane (not necessarily X), but have not taken exactly the same route within that time. Determine all possible values of q.

Walkthrough —

(a) Let q be a positive rational number such that the two ants can be at the same point in the plane after a whole number of minutes, without taking the same route.

(b) Let r be a positive integer such that the two ants are at the same point in the plane after r minutes, without taking the same route.

(c) Prove that

$$\varepsilon_1 q + \varepsilon_2 q^2 + \cdots + \varepsilon_r q^r = \varepsilon_1' q + \varepsilon_2' q^2 + \cdots + \varepsilon_r' q^r,$$

for some $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$, $\varepsilon_1', \varepsilon_2', \ldots, \varepsilon_r'$ are elements of $\{1, -1, i, -i\}$, and the tuples $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$ and $(\varepsilon_1', \varepsilon_2', \ldots, \varepsilon_r')$ are not equal.

(d) Prove that there exists a nonzero polynomial P of degree at most $r-1$, which vanishes at q, and has coefficients in $\{0, 2, 1+i\}$ up to a factor of $\pm 1, \pm i$.

(e) Further, prove that there exists a nonzero polynomial Q of degree at most $r-1$, which vanishes at q, and has coefficients in $\{0, 1, 1+i\}$ up to a factor of $\pm 1, \pm i$.

(f) Write $q = \frac{a}{b}$ where a, b are relatively prime positive integers, and show that b divides the leading coefficient of Q in $\mathbb{Z}[i]$, and a divides the constant term of Q in $\mathbb{Z}[i]$, where

$$\mathbb{Z}[i] = \{x + yi : x, y \in \mathbb{Z}\}$$

is the ring of Gaussian integers.

(g) Prove further that a divides the constant term of Q in $\mathbb{Z}[i]$.

(h) Conclude that $q = 1$ is the only possible value of q.

Exercise 13 (Brazil National Olympiad 2018 Level 2 P5,). Initially, the numbers $1, 2, 3, \ldots, 10$ are written on a board. An operation consists of choosing two numbers a and b from the board, deleting them, and writing the number $a + b + \frac{ab}{f(a,b)}$, where $f(a, b)$ denotes the sum of all numbers currently on the board excluding a and b. This process is repeated until only two numbers x and y remain, with $x \geq y$. What is the maximum possible value of x?

Walkthrough — Show that the sum of the product of the pairs of the numbers written on the board is an invariant.

Exercise 14 (Brazil National Olympiad 2022 Level 3 P1,). A single player game has the following rules: initially, there are 10 piles of stones with $1, 2, \ldots, 10$ stones, respectively. A movement consists of making one of the following operations:

(i) to choose 2 piles, both of them with at least 2 stones, combine them and then add 2 stones to the new pile;

(ii) to choose a pile with at least 4 stones, remove 2 stones from it, and then split it into two piles with amount of stones in those two piles to be chosen by the player.

The game continues until it is not possible to make an operation. Show that the number of piles with one stone in the end of the game is always the same, no matter how the movements are made.

Walkthrough —

(a) Let p denote the number of piles and s denote the total number of stones at some stage of the game.

(b) Note that after applying operation (i), (p, s) changes to $(p-1, s+2)$, and after applying operation (ii), (p, s) changes to $(p+1, s-2)$.

(c) Construct a polynomial in p and s using the above observations, which remains invariant under both the operations.

Remark. Note that applying the second operation to the pile obtained by applying the first operation, may yield the piles, to which the first operation was applied. Thus the game need not terminate. This problem seems to ask that the number of piles with one stone in the end of the game (when an end state is reached) is always the same, no matter how the movements are made.

Exercise 15 (Belarus National Olympiad 2022 Grade 9 Day 1 P4,). The numbers $1, 2, \ldots, 50$ are written on the board. Anya does the following operation: removes the numbers a and b from the board and writes their sum $a+b$, after which she also notes down the number $ab(a+b)$. After 49 of these operations, only one number is left on the board. Anya summed up all the 49 numbers in her notes and got S.

(a) Prove that S does not depend on the order of Anya's actions.

(b) Calculate S.

Walkthrough —

(a) Let us denote the sum of the cubes of the numbers on the board at a moment as B, and the sum of the numbers in Anya's notes at the same moment as N.

(b) Prove that $B - 3N$ is invariant under Anya's operation.

(c) Calculate S using the invariance of $B - 3N$.

Exercise 16 (British Mathematical Olympiad Round 2 2023 P2). For an integer $n > 1$, the numbers $1, 2, 3, \ldots, n$ are written in order on a blackboard. The following *moves* are possible:

(i) Take three adjacent numbers x, y, z whose sum is a multiple of 3 and replace them with y, z, x.

(ii) Take two adjacent numbers x, y whose difference is a multiple of 3 and replace them with y, x.

For example we could take: $1,2,3,4 \xrightarrow{\text{(i)}} 2,3,1,4 \xrightarrow{\text{(ii)}} 2,3,4,1$. Find all n such that the initial list can be transformed into $n, 1, 2, \ldots, n-1$ after a finite number of moves.

Walkthrough —

(a)

Exercise 17 (Chip-firing game)**.** Assume that on the Cartesian plane, four chips are placed at the origin. In each step, you may choose a lattice point (x, y) having at least a chip on it, remove one chip from (x, y), and place one chip each at $(x+1, y)$ and one at $(x, y+1)$. Show that after a finite number of steps, there are at least two chips at some lattice point.

Remark. The problem seems to ask that after any number of finitely many moves, there will exist two coins placed at the same point.

Walkthrough —

(a)

Exercise 18 (Serbia IMO TST 2024 Day 1 P1,)**.** Three coins are placed at the origin of a Cartesian coordinate system. On one move one removes a coin placed at some position (x, y) and places three new coins at $(x+1, y)$, $(x, y+1)$ and $(x+1, y+1)$. Prove that after finitely many moves, there will exist two coins placed at the same point.

Remark. The problem seems to ask that after any number of finitely many moves, there will exist two coins placed at the same point.

Walkthrough —

(a)

Exercise 19 (St Petersburg Mathematical Olympiad 2025 Round 2 Grade 11 P1,)**.** 100 red and 99 blue piranhas were released into the empty Quiet Pool. When a piranha wants to eat, it can eat two other piranhas. If its victims are of the same color, the piranha changes its color, but if they are multicolored, it does not change. In the end, only one piranha remains. What color is it?

Walkthrough —

(a) Show that the number of blue piranhas and the number of red piranhas differ by a multiple of 4.

(b) Show that a red piranha remains at the end of the process.

Remark. We can ask to determine the colors of the remaining piranhas at the end, if initially there were 100 red and 100 blue piranhas.

Exercise 20 (Dutch Mathematical Olympiad 2025 P3,). Amber has a square consisting of 10×10 square boxes. One at a time, she randomly chooses one of the hundred squares and fills it with a number. The number she puts in a square is always equal to the total number of squares already filled in that square's row and column. So in the first square she chooses she writes a 0, in the second square a 0 or a 1, depending on whether it is in a different row and a different column or not, et cetera. So in each square there will be a number from 0 to 18 inclusive. After Amber has written a number in each square, she adds up all the numbers. What is/are the possible result(s) of this addition?

Walkthrough —

(a) In a chosen square, instead of writing the sum of the number of filled squares in the row and column, let us write down the pair of numbers (r, c) where r is the number of filled squares in that row before filling the square, and c is the number of filled squares in that column before filling the square.

(b) It amounts to find the sum of the numbers of the form $r + c$ across all the squares.

(c) Try to sum up the r values across all the squares, and then sum up the c values across all the squares.

(d) Show that the result of the addition is always the same, regardless of the order in which Amber fills the squares.

Exercise 21 (RMO 2016d P4,). A box contains 4032 answer scripts out of which exactly half have odd number of marks. We choose 2 scripts randomly and, if the scores on both of them are odd number, we add one mark to one of them, put the script back in the box and keep the other script outside. If both scripts have even scores, we put back one of the scripts and keep the other outside. If there is one script with even score and the other with odd score, we put back the script with the odd score and keep the other script outside. After following this procedure a number of times, there are 3 scripts left among which there is at least one script each with odd and even scores. Find, with proof, the number of scripts with odd scores among the three left.

> **Walkthrough** — What happens to the number of scripts with odd scores?

Exercise 22 (). Consider the sequence of integers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots,$$

where the first two terms are 1, and each subsequent term is the sum of the previous two terms. Prove that for any positive integer n, the greatest common divisor of the n-th and $(n+1)$-st terms is 1.

> **Walkthrough** — Show that the greatest common divisor of the n-th and $(n+1)$-st terms is an invariant of the sequence.

§2 Strategies

Exercise 23 (Covering a mutilated chessboard). Assume that from an 8×8 chessboard, two squares of different colors are removed. Prove that the remaining 62 squares can be covered by 31 dominoes, where each domino covers exactly two adjacent squares.

> **Walkthrough** — Show that there is a trail passing through all the squares of the chessboard, passing through each square exactly once, such that the adjacent squares in the trail are of different colors.

Exercise 24 (China TST 1995 Day 2 P2). Alice and Bob play a game with a polynomial of degree at least 4:

$$x^{2n} + \square x^{2n-1} + \square x^{2n-2} + \dots + \square x + 1.$$

They take turns to fill the empty boxes. If the resulting polynomial has no real root, Alice wins, otherwise, Bob wins. If Alice goes first, who has a winning strategy?

> **Walkthrough** —
>
> **(a)** There are more odds than evens among the integers $1, 2, \dots, 2n-1$.
>
> **(b)** Can Bob have a winning strategy using the odds in his favour?

Exercise 25 (British Mathematical Olympiad Round 1 2003/4 P3). Alice and Barbara play a game with a pack of $2n$ cards, on each of which is written a positive integer. The pack is shuffled and the cards laid out in a row, with the numbers facing upwards. Alice starts, and the girls take turns to remove one card from either end of the row, until Barbara picks up the final card. Each girl's score is the sum of the numbers on her chosen cards at the end of the game. Prove that Alice can always obtain a score at least as great as Barbara's.

Walkthrough —

(a)

Exercise 26 (Moldova National Olympiad 2006 Grade 11 P4, cf. QEDMO 2006 P11)**.** On a table, there are 2006 cards, each with a natural number written on it. Two players play a game, taking turns. In each turn, a player takes a card from either the leftmost or the rightmost end of the row of cards. The game ends when all cards are taken. After the game, each player calculates the sum of the numbers on the cards they have taken. If the sum of the first player is not less than the sum of the second player, then the first player wins. Show that the first player has a winning strategy. (Note that the integers on the cards are visible to both players throughout the game.)

Walkthrough —

(a) Show that the first player can always take the cards in the odd positions or the cards in the even positions.

(b) Before the game starts, the first player compares the sum of the numbers on the cards in the odd positions and the sum of the numbers on the cards in the even positions.

Exercise 27 (Bulgaria National Olympiad 2010 Day 1 P1)**.** A table 2×2010 is divided to unit cells. Ivan and Peter are playing the following game. Ivan starts, and puts horizontal 1×2 domino that covers exactly two unit table cells. Then Peter puts vertical 2×1 domino that covers exactly two unit table cells. Then Ivan puts horizontal 1×2 domino. Next Peter puts vertical 2×1 domino, and so on. The player who cannot put a domino loses. Which player has a winning strategy?

Walkthrough —

(a)

Exercise 28 (Junior Balkan MO Shortlist 2015 C1)**.** An $n \times n$ board (with $n \geq 3$) is divided into n^2 unit squares. Integers from 0 to n included, are written down: one integer in each unit square, in such a way that the sums of integers in each 2×2 square of the board are different. Find all n for which such boards exist.

Walkthrough —

(a)

0	0	0
0	0	1
1	2	3

(a)

0	1	2
0	0	0
0	2	2

(b)

0	1	2
0	0	0
0	0	2

(c)

Figure 1: Junior Balkan MO Shortlist 2015 C1, Exercise 28, for $n = 3$

0	0	0	0
0	0	1	1
1	2	2	3
1	0	2	1

(a)

4	4	4	4
0	0	2	2
0	0	0	0
0	1	2	3

(b)

Figure 2: Junior Balkan MO Shortlist 2015 C1, Exercise 28, for $n = 4$

5	5	5	5	5
0	0	2	2	4
0	0	0	0	0
0	1	2	3	4
5	5	5	5	5

(a)

6	6	6	6	5	5
6	6	5	5	5	5
0	0	2	2	4	4
0	0	0	0	0	0
0	1	2	3	4	5
6	3	5	5	5	5

(b)

Figure 3: Junior Balkan MO Shortlist 2015 C1, Exercise 28, for $n = 5$ and $n = 6$

Exercise 29 (European Mathematical Cup 2016 Senior Category P2, proposed by Petar Orlić)**.** For two positive integers a and b, Ivica and Marica play the following game: Given two piles of a and b cookies, on each turn a player takes $2n$ cookies from one of the piles (with $n \geq 1$), of which he eats n and puts n of them on the other pile. Number n is arbitrary in every move. Players take turns alternatively, with Ivica going first. The player who cannot make a move, loses. Assuming both players play perfectly, determine all pairs of numbers (a, b) for which Marica has a winning strategy.

It is worth noting the apparent similarity of the above problem to Exercise 33.

> **Walkthrough** —
>
> **(a)**

Exercise 30 (Canadian Mathematical Olympiad 2016 P1)**.** The integers $1, 2, 3, \ldots, 2016$ are written on a board. You can choose any two numbers on the board and replace them with their average. For example, you can replace 1 and 2 with 1.5, or you can replace 1 and 3 with a second copy of 2. After 2015 replacements of this kind, the board will have only one number left on it.

(i) Prove that there is a sequence of replacements that will make the final number equal to 2.

(ii) Prove that there is a sequence of replacements that will make the final number equal to 1000.

> **Walkthrough** —
>
> **(a)**

Exercise 31 (RMO 2016e P6)**.** A deck of 52 cards is given. There are four suites each having cards numbered $1, 2, \ldots, 13$. The audience chooses some five cards with distinct numbers written on them. The assistant of the magician comes by, looks at the five cards and turns exactly one of them face down and arranges all five cards in some order. Then the magician enters and with an agreement made beforehand with the assistant, he has to determine the face down card (both suite and number). Explain how the trick can be completed.

Exercise 32 (SMMC 2017 A1)**.** The five sides and five diagonals of a regular pentagon are drawn on a piece of paper. Two people play a game, in which they take turns to colour one of these ten line segments. The first player colours line segments blue, while the second player colours line segments red. A player cannot colour a line segment that has already been coloured. A player wins if they are the first to create a triangle in their own colour, whose three vertices are also vertices of the regular pentagon. The game is declared a draw if all ten line segments have been coloured without a player winning. Determine whether the first player, the second player, or neither player can force a win.

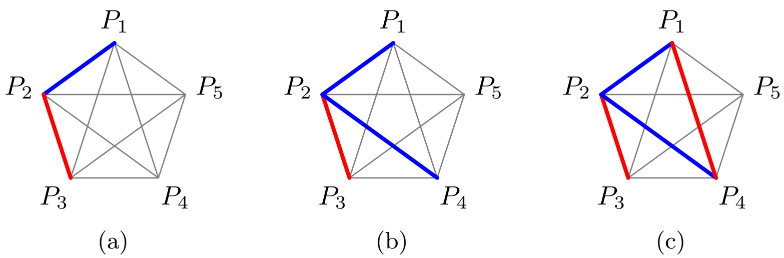


Figure 4: SMMC 2017 A1, Exercise 32, First two colored segments share a common vertex

Walkthrough — To force a win, the first player colors a side of the pentagon in blue in the first move.

Let us first consider the case that the two edges coloured in the first two moves have a common endpoint, as in Fig. 4.

(a) Show that there is no loss of generality in assuming that the first player colours P_1P_2 in blue and the second player colours P_2P_3 in red, where $P_1P_2P_3P_4P_5$ denotes the pentagon.

(b) Next, the first player colours P_2P_4 in blue.

(c) If the second player does not color P_1P_4, then in the next move the first player colours P_1P_4 in blue and wins. It remains to consider the case that the second player colours P_1P_4 in red, which we assume from now on.

(d) The first player colours P_2P_5 in blue.

(e) During the next turn of the first player, one of the edges P_1P_5 and P_4P_5 is coloured in blue, thus creating a blue triangle.

We are still left with the case that the two edges coloured in the first two moves do not have a common endpoint, as in Fig. 5.

(a) Show that there is no loss of generality in assuming that the first player colours P_1P_2 in blue and the second player colours P_3P_4 in red.

(b) Next, the first player colours P_1P_5 in blue.

(c) If the second player does not color P_2P_5, then in the next move the first player colours P_2P_5 in blue and wins. It remains to consider the case that the second player colours P_2P_5 in red, which we assume from now on.

(d) The first player colours P_1P_3 in blue.

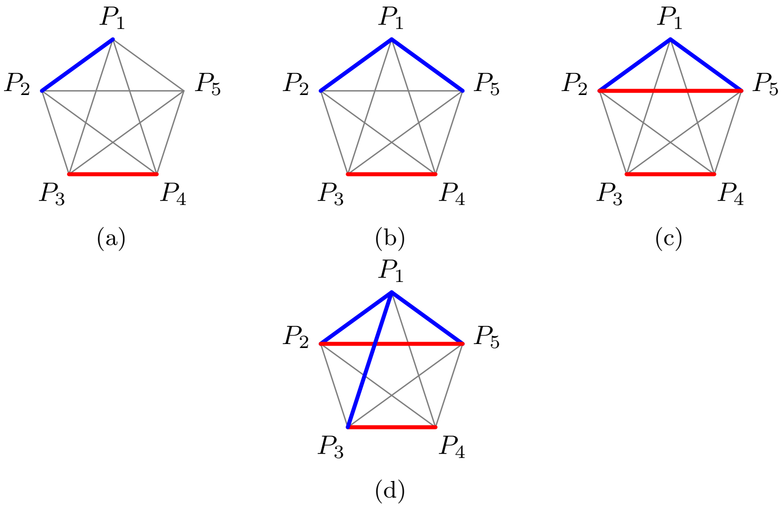


Figure 5: SMMC 2017 A1, Exercise 32, First two colored segments share no common vertex

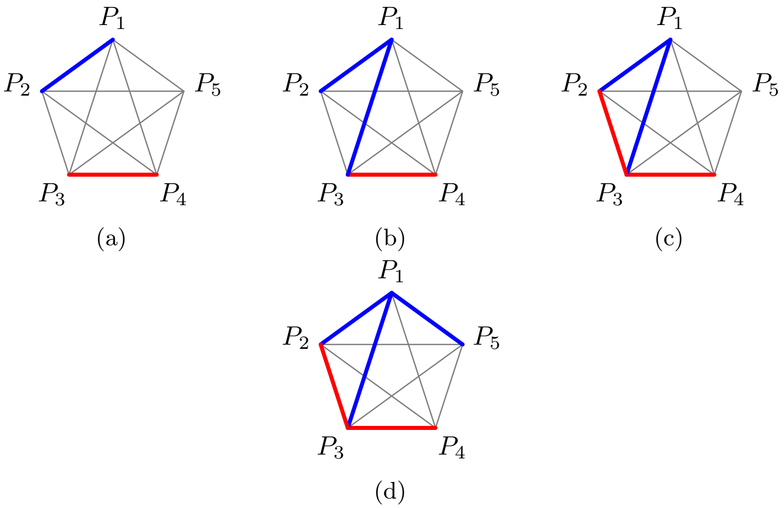


Figure 6: SMMC 2017 A1, Exercise 32, First two colored segments share no common vertex

(e) During the next turn of the first player, one of the edges P_2P_3 and P_3P_5 is coloured in blue, thus creating a blue triangle.

Here is another strategy to see that the first player can force a win, if the two edges coloured in the first two moves do not have a common endpoint, as in Fig. 6.

(a) Show that there is no loss of generality in assuming that the first player colours P_1P_2 in blue and the second player colours P_3P_4 in red.

(b) Next, the first player colours P_1P_3 in blue.

(c) This forces the second player to colour P_2P_3 in red, otherwise the first player colours P_2P_3 in blue and wins.

(d) The first player colours P_1P_5 in blue.

(e) During the next turn of the first player, one of the edges P_2P_5 and P_3P_5 is coloured in blue, thus creating a blue triangle.

Exercise 33 (Balkan Mathematical Olympiad 2018 P3, , proposed by Dimitris Christophides, Cyprus)**.** Alice and Bob play the following game: They start with non-empty piles of coins. Taking turns, with Alice playing first, each player choose a pile with an even number of coins and moves half of the coins of this pile to the other pile. The game ends if a player cannot move, in which case the other player wins.

Determine all pairs (a, b) of positive integers such that if initially the two piles have a and b coins respectively, then Bob has a winning strategy.

It is worth noting the apparent similarity of the above problem to Exercise 29.

Walkthrough —

(a)

Exercise 34 (Mexico National Olympiad 2020 P3, , proposed by Victor and Isaías de la Fuente)**.** Let $n \geq 3$ be an integer. Two players, Ana and Beto, play the following game. Ana tags the vertices of a regular n-gon with the numbers from 1 to n, in any order she wants. Every vertex must be tagged with a different number. Then, we place a turkey in each of the n vertices. These turkeys are trained for the following. If Beto whistles, each turkey moves to the adjacent vertex with greater tag. If Beto claps, each turkey moves to the adjacent vertex with lower tag. Beto wins if, after some number of whistles and claps, he gets to move all the turkeys to the same vertex. Ana wins if she can tag the vertices so that Beto can't do this. For each $n \geq 3$, determine which player has a winning strategy.

Walkthrough —

(a)

Exercise 35 (Mexico National Olympiad 2020 P4, proposed by Victor de la Fuente)**.** Let $n \geq 3$ be an integer. In a game there are n boxes in a circular array. At the beginning, each box contains an object which can be rock, paper or scissors, in such a way that there are no two adjacent boxes with the same object, and each object appears in at least one box. Same as in the game, rock crushes scissors, scissors cut paper, and paper covers rock. The game consists of moving objects from one box to another according to the following rule:

> Two adjacent boxes and one object from each one are chosen in such a way that these are different, and we move the loser object to the box containing the winner object. For example, if we picked rock from box A and scissors from box B, we move scissors to box A.

Prove that, applying the rule enough times, it is possible to move all the objects to the same box.

Walkthrough —

(a)

Exercise 36 (Canadian Junior Mathematical Olympiad 2022 P2)**.** You have an infinite stack of T-shaped tetrominoes (composed of four squares of side length 1), and an $n \times n$ board. You are allowed to place some tetrominoes on the board, possibly rotated, as long as no two tetrominoes overlap and no tetrominoes extend off the board. For which values of n can you cover the entire board?

Walkthrough —

(a)

Exercise 37 (Junior Balkan MO 2023 P3, proposed by Théo Lenoir, France)**.** Alice and Bob play the following game on a 100×100 grid, taking turns, with Alice starting first. Initially the grid is empty. At their turn, they choose an integer from 1 to 100^2 that is not written yet in any of the cells and choose an empty cell, and place it in the chosen cell. When there is no empty cell left, Alice computes the sum of the numbers in each row, and her score is the maximum of these 100 numbers. Bob computes the sum of the numbers in each column, and his score is the maximum of these 100 numbers. Alice wins

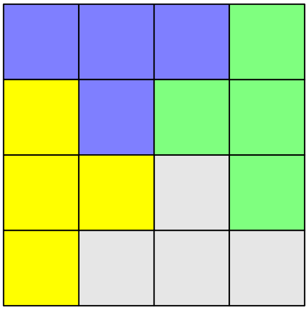


Figure 7: Canada Junior Mathematical Olympiad 2022 P2, Exercise 36

if her score is greater than Bob's score, Bob wins if his score is greater than Alice's score, otherwise no one wins.

Find if one of the players has a winning strategy, and if so which player has a winning strategy.

Walkthrough —

(a)

Remark. Dragomir Grozev argues that for the analogous problem on an $n \times n$ grid with n odd, Alice has a winning strategy.

Exercise 38 (Swiss Mathematical Olympiad 2023 Final Round Day 1 P2,). The wizard Albus and Brian are playing a game on a square of side length $2n + 1$ meters surrounded by lava. In the centre of the square there sits a toad. In a turn, a wizard chooses a direction parallel to a side of the square and enchants the toad. This will cause the toad to jump d meters in the chosen direction, where d is initially equal to 1 and increases by 1 after each jump. The wizard who sends the toad into the lava loses. Albus begins and they take turns. Depending on n, determine which wizard has a winning strategy.

Walkthrough —

(a)

Exercise 39 (Columbia National Olympiad 2024 P1, , proposed by Juan David Restrepo). Daniel and Juan David have three piles of stones consisting of 30 yellow stones, 20 blue stones and 10 red stones respectively. They then take turns to remove one stone from any pile, starting with Daniel. Daniel wins if at any moment there are two piles with the same positive number of stones; in any other case, Juan David wins. Determine which of the two players has a winning strategy and describe it.

Walkthrough —

(a) Show that if Daniel removes a yellow stone during his turns, then he will win.

(b) Assume that after k turns in total, only one yellow stone is left.

(c) Assume that among these k turns, Daniel has removed x yellow stones, and Juan David has removed y yellow stones, b blue stones and r red stones.

(d) Show that $b + r \leq b + r + y \leq x \leq x + y = 29$.

(e) Prove that the difference between the number of yellow stones and the number of blue stones or the difference between the number of yellow stones and the number of red stones is less than or equal to zero after k turns.

(f) Conclude that Daniel wins.

Exercise 40 (Columbia Junior Mathematical Olympiad 2024 P3, proposed by Santiago Rodriguez)**.** Let $n > 1$ be a positive integer. There are n rocks arranged in a circle, with one frog sitting on each rock. The numbers $1, 2, \ldots, n$ are written on the rocks in some order. After a signal is given, each frog looks at the number i written on the rock it occupies and then jumps i positions clockwise along the circle. Determine all values of n for which it is possible to assign the numbers $1, 2, \ldots, n$ to the rocks in such a way that, after the signal, no two frogs land on the same rock.

Walkthrough —

(a) Label the rocks as the first, second, ..., n-th rock in the clockwise order. For each $i \in \{1, 2, \ldots, n\}$, let a_i be the number written on the i-th rock.

(b) Observe that if n is a positive integer such that there is an assignment of the numbers $1, 2, \ldots, n$ to the rocks such that no two frogs land on the same rock after the signal, then the numbers $i + a_i$ for $i = 1, 2, \ldots, n$ are distinct modulo n.

(c) Show that integer n satisfying the given condition must be odd.

(d) Show that if n is odd, then there is an assignment of the numbers $1, 2, \ldots, n$ to the rocks such that no two frogs land on the same rock after the signal.

Exercise 41 (Taiwan TST 2024 Round 1 Quiz 1 proposed by Ho-Chien Chen)**.** Let $n \geq 5$ be a positive integer. There are n stars with values 1 to n, respectively. Anya and Becky play a game. Before the game starts, Anya places the n stars in a row in whatever order she wishes. Then, starting from Becky, each player takes the left-most or right-most star in the row. After all the stars have been taken, the player with the highest total value of stars wins;

if their total values are the same, then the game ends in a draw. Find all n such that Becky has a winning strategy.

Walkthrough —

(a)

Exercise 42 (Mexico National Olympiad 2024 P6,). Ana and Beto play on a blackboard where all integers from 1 to 2024 (inclusive) are written. On each turn, Ana chooses three numbers a, b, c written on the board and then Beto erases them and writes one of the following numbers:

$$a + b - c, a - b + c, -a + b + c.$$

The game ends when only two numbers are left on the board and Ana cannot play. If the sum of the final numbers is a multiple of 3, Beto wins. Otherwise, Ana wins. Who has a winning strategy?

Walkthrough —

(a) Can Beto have a strategy to keep the parity of the number of integers congruent to i modulo 3 unchanged for any $i \in \{0, 1, 2\}$?

(b) Note that if Ana chooses three numbers which are congruent to $0, 1, 2$ modulo 3 in some order, then Beto will fail to do so, and otherwise Beto keep these parities unchanged.

(c) If Ana makes such a move, then Beto can write the integer among the three choices that is divisible by 3.

(d) For the next moves, can Beto follow some strategy to keep the parity of the number of integers congruent to i modulo 3 unchanged for any $i \in \{0, 1, 2\}$, and keep at least two integers divisible by 3 to ensure winning the game?

(e) Note that if Ana chooses three numbers which are congruent to any of the unordered triples $(0, 1, 2)$, $(0, 0, 1)$, $(0, 0, 2)$ modulo 3 in some order, then Beto will again fail to do so, and otherwise Beto can have a strategy as sought in the previous step.

(f) If Ana makes such a move, then does Beto have a strategy to reduce the game to a collection of an even number of integers congruent to 0 modulo 3, an odd number of integers congruent to 1 modulo 3, and an odd number of integers congruent to 2 modulo 3, similar to the initial configuration of the game, and then start afresh to win the game by induction?

Exercise 43 (Dutch IMO TSTST 2024 P4,). Let n be a positive integer with $n \geq 3$. Consider a board of $n \times n$ boxes. In each step taken, the colors of the 5 boxes that make up the figure below change color (black boxes change to white and white boxes change to black). The figure can be rotated $90°, 180°$ or $270°$. Initially, all the boxes are white. Determine for what values of n it can be achieved, through a series of steps, that all the squares on the board are black.

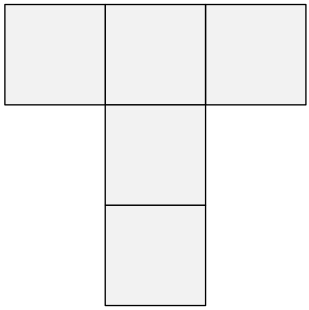


Figure 8: Dutch IMO TSTST 2024 P4, Exercise 43

Walkthrough —

(a)

Exercise 44 (Austrian Mathematical Olympiad, National Competition, Preliminary Round 2025 P3, , proposed by Michael Reitmeir)**.** Consider the following game for a positive integer n. Initially, the numbers $1, 2, \ldots, n$ are written on a board. In each move, two numbers are selected such that their difference is also present on the board. This difference is then erased from the board. (For example, if the numbers $3, 6, 11$ and 17 are on the board, then 3 can be erased as $6 - 3 = 3$, or 6 as $17 - 11 = 6$, or 11 as $17 - 6 = 11$.)

For which values of n, is it possible to end with only one number remaining on the board?

Walkthrough — Assume that it is possible to end with only one number remaining on the board.

(a) Show that if d is an odd integer dividing the integers present on the board at some stage, then d divides the integer that was erased in the move prior to this stage, that is, d divides all the integers present on the board prior to this stage as well.

(b) Show that n has no odd positive divisor other than 1.

Next, assume that n is a power of 2. Use induction to show that it is possible to end with only one number remaining on the board.

Exercise 45 (Rioplatense Mathematical Olympiad 2025 Level 1 Grade 8-9 P3,)**.** Liko plays a game called The 24-N cards. In this game, N cards are placed on the table, each showing an integer between 1 and 9, inclusive. Some cards may have the same number. Liko wins if she can write a mathematical expression equal to 24, using each of the cards exactly once. She may only use the operations addition, subtraction, multiplication, and division, and she may group the operations in any way she wants, but multiplying by 0 is not allowed in this game. For instance, if the cards on the table are $\{7, 1, 7, 2\}$, then Liko can win by forming

$$(7 \times 7 - 1) \div 2 = 24.$$

Determine whether there exists a number N such that, regardless of what numbers appear on the N cards, Liko can always win.

Walkthrough —

(a)

Exercise 46 (Brazil National Olympiad 2025 Level 2 P2,). Let m be a positive integer. Ana and Banana play the following game on a 5×5 board, which is initially filled with 0's in all squares. Taking turns starting with Ana, they choose a square on the board and add an integer from the set $\{1, 2, 3, 4, 5\}$ to the number currently in that square, such that the number in the square does not exceed m. The winner is the player who makes a move that completes a row, a column, or one of the two main diagonals with the number m in all five squares. For which positive integers m does Ana, the first player, have a winning strategy?

Walkthrough —

(a)

Exercise 47 (Columbia National Olympiad 2025 P2, , proposed by Santiago Rodriguez). The numbers $1, 2, \ldots, 2025$ are written around a circle in some order. On each turn, Celeste chooses an integer $1 \leq n \leq 2025$. Then she selects the number n and the next $n-1$ numbers following it in the clockwise direction and inverts their order on the circle. Prove that, after a finite amount of turns, Celeste can put the numbers on the circle in the order $1, 2, 3, \ldots, 2025$ in the clockwise direction, regardless of their original arrangement on the circle.

Walkthrough —

(a) Show that if the numbers $2, 3, x$ appear consecutively in that order in the clockwise direction, then after a finite number of turns, they can be arranged in the order $x, 2, 3$ in the clockwise direction, while keeping the order of the other numbers unchanged.

(b) Show that if the numbers a, b appear consecutively in that order in the clockwise direction, then after a finite number of turns, they can be arranged in the order b, a in the clockwise direction, while keeping the order of the other numbers unchanged.

(c) Using the above two results, prove that, given any arrangement of the numbers $1, 2, \ldots, 2025$ around the circle, after a finite number of turns, they can be arranged in the order $1, 2, 3, \ldots, 2025$ in this order in the clockwise direction.

Here is an alternative solution to the problem.

Exercise 48 (All-Russian Mathematical Olympiad 2025 Grade 9 Day 2 P7, , proposed by I. A. Efremov)**.** The numbers $1, 2, 3, \ldots, 60$ are written in a row in that exact order. Igor and Ruslan take turns inserting the signs $+, -, \times$ between them, starting with Igor. Each turn consists of placing one sign. Once all signs are placed, the value of the resulting expression is computed. If the value is divisible by 3, Igor wins; otherwise, Ruslan wins. Which player has a winning strategy regardless of the opponent's moves?

Walkthrough —

(a) Igor, the first player, has a winning strategy.

(b) Note that the numbers $1, 2, 3, \ldots, 30$ are congruent to $31, 32, 33, \ldots, 60$ modulo 3, respectively.

(c) Igor in his first turn places a $-$ sign between 30 and 31.

(d) After that, the numbers $1, 2, 3, \ldots, 30$ can be thought as grouped together and the numbers $31, 32, 33, \ldots, 60$ can also be thought as grouped together.

(e) For each choice of Ruslan, Igor can respond by placing the corresponding sign in the other half.

Exercise 49 (Cono Sur Olympiad 2025 P4,)**.** Lucero and Pablo play a game on the board shown in Fig. 9. Lucero plays first, and they take turns. In each turn, a player chooses an unpainted circle from the bottom row and paints it blue, green, or red. This continues for four turns until the entire bottom row is painted.

Then, the rest of the board is painted according to the following rules:

(i) If two adjacent circles in a row are the same color, the circle above and adjacent to them is painted that same color.

(ii) If two adjacent circles in a row are of different colors, the circle above and adjacent to them is painted with the third color.

This procedure is repeated until all the circles on the board are painted. Lucero wins if the single top circle is painted red or green, and Pablo wins if it is painted blue.

Determine who has a winning strategy.

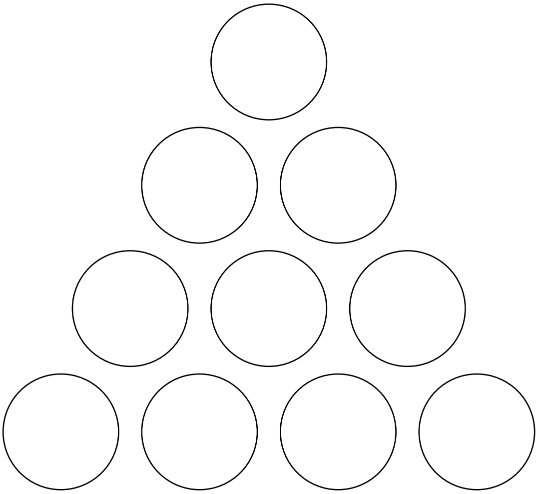


Figure 9: Cono Sur Olympiad 2025 P4, Exercise 49

Walkthrough —

(a) Use the congruence classes modulo 3 to represent the colors of the circles. For example, we can represent blue as 0, green as 1, and red as -1 modulo 3.

(b) Let a, b, c, d be the colors of the circles in the bottom row from left to right, respectively, represented as congruence classes modulo 3.

(c) Express the color of the top circle as a function of the colors of the circles in the bottom row.

(d) Determine a winning strategy for Pablo based on the colors of the circles in the bottom row.

Exercise 50 (Singapore Junior Mathematical Olympiad 2025 P3,). Jack and Jill play the following game: Jack throws 3 dice and Jill can select some of them, possibly none, and turn each of them to the opposite side. Jill wins if the sum of the values on the dice is a multiple of 4. Can Jill always win? (Note the game is played with standard dice where the sum of the numbers on opposite sides is 7.)

Walkthrough —

(a)

§3 Optimization problems

Exercise 51 (Princeton University Mathematics Competition 2009 Combinatorics A6,). We have a 6×6 square, partitioned into 36 unit squares. We select some of these unit squares and draw some of their diagonals, subject to the condition that no two diagonals we draw have any common points. What is the maximal number of diagonals that we can draw?

> **Walkthrough** —
> **(a)**

> **Remark.** See also this on AoPS.

Exercise 52 (Tournament of Towns 2014 Spring Junior A Level P2, , by A. V. Shapovalov). Peter marks several cells on a 5×5 board. Basil wins if he can cover all marked cells with three-cell corners. The corners must be inside the board and not overlap. What is the least number of cells Peter should mark to prevent Basil from winning? (Cells of the corners must coincide with the cells of the board).

> **Walkthrough** —
> **(a)**

Exercise 53 (Tournament of Towns 2014 Spring Senior A Level P3, by I. I. Bogdanov). The King called two wizards. He ordered First Wizard to write down 100 positive real numbers (not necessarily distinct) on cards without revealing them to Second Wizard. Second Wizard must correctly determine all these numbers, otherwise both wizards will lose their heads. First Wizard is allowed to provide Second Wizard with a list of distinct numbers, each of which is either one of the numbers on the cards or a sum of some of these numbers. He is not allowed to tell which numbers are on the cards and which numbers are their sums. Finally the King tears as many hairs from each wizard's beard as the number of numbers in the list given to Second Wizard. What is the minimal number of hairs each wizard should lose to stay alive?

> **Walkthrough** —
> **(a)**

Exercise 54 (Junior Balkan MO 2015 P4,). An L-shape is one of the following four pieces, each consisting of three unit squares:

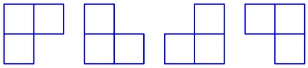


Figure 10: Junior Balkan MO 2015 P4, Exercise 54

A 5×5 board, consisting of 25 unit squares, a positive integer $k \leq 25$ and an unlimited supply of L-shapes are given. Two players A and B, play the following game: starting with A they play alternatively mark a previously unmarked unit square until they marked a total of k unit squares.

A placement of L-shapes on unmarked unit squares is called *good* if the L-shapes do not overlap and each of them covers exactly three unmarked unit squares of the board.

B wins if every good placement of L-shapes leaves uncovered at least three unmarked unit squares. Determine the minimum value of k for which B has a winning strategy.

Exercise 55 (Moscow Mathematical Olympiad 2018 Grade 11 Day 2 P2, , proposed by M. A. Evdokimov)**.** A number of dominoes were placed without overlapping on a 2018×2018 square board, each covering exactly two squares. It turned out that no two dominoes share a common whole side, that is, no two dominoes form either a 2×2 square or a 4×1 rectangle. Is it possible to cover more than 99% of all the squares on the board?

> **Walkthrough** —
>
> **(a)**

Exercise 56 (Moscow Mathematical Olympiad 2018 Grade 11 Day 2 P5, , proposed by P. A. Borodin)**.** Zhenya painted a spherical egg sequentially using five different dyes, dipping it into a glass containing the current dye in such a way that exactly half of the egg's surface area (a hemisphere) was colored. As a result, the egg became completely colored. Prove that one of the dyes was superfluous, that is, if Zhenya had omitted that particular dye but had dipped the egg into the remaining dyes in the exact same manner, the egg would still have ended up completely colored.

> **Walkthrough** —
>
> **(a)**

> **Remark.** We refer to Carathéodory's theorem on convex hulls.

Exercise 57 (Benelux Mathematical Olympiad 2020 P2,)**.** Let N be a positive integer. A collection of $4N^2$ unit tiles with two segments drawn on them as shown in Fig. 11 is assembled into a $2N \times 2N$ board. Tiles can be rotated.

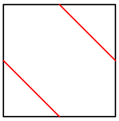


Figure 11: BxMO 2020 P2, Exercise 57

The segments on the tiles define paths on the board. Determine the least possible number and the largest possible number of such paths.

For example, there are 9 *paths on the* 4×4 *board shown in Fig. 12.*

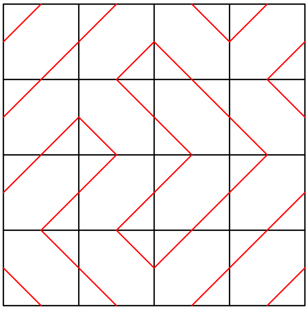


Figure 12: BxMO 2020 P2, Exercise 57

Walkthrough —

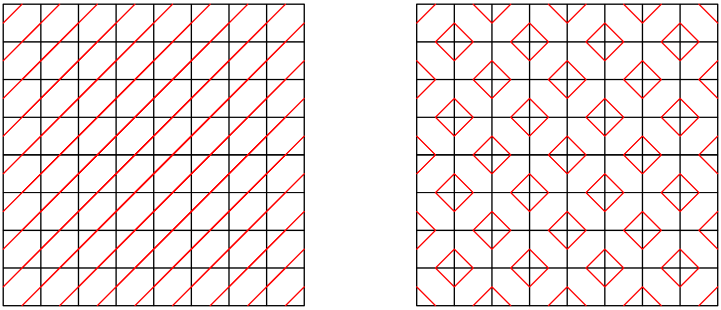
(a)

Exercise 58 (Swiss Mathematical Olympiad 2023 Final Round Day 2 P8,). Let n be a positive integer. We start with n piles of pebbles, each initially containing a single pebble. One can perform moves of the following form: choose two piles, take an equal number of pebbles from each pile and form a new pile out of these pebbles. Find (in terms of n) the smallest number of nonempty piles that one can obtain by performing a finite sequence of moves of this form.

Walkthrough —

(a)

Exercise 59 (Kosovo National Olympiad 2024 Grade 8 P3,). The numbers $1, 2, 3, \dots, 20$ are placed in any order in a circle. A number is considered to be



(a) (b)

Figure 13: BxMO 2020 P2, Exercise 57

high if it is larger than the two numbers near it. What is the greatest possible number of high numbers?

Walkthrough —

(a) Prove that there are at most 10 high numbers in any arrangement of $1, 2, \ldots, 20$ on a circle.

(b) Find an arrangement of $1, 2, \ldots, 20$ on a circle having 10 high numbers.

Exercise 60 (Junior Balkan MO 2019 P4,). A 5×100 table is divided into 500 unit square cells, where n of them are coloured black and the rest are coloured white. Two unit square cells are called adjacent if they share a common side. Each of the unit square cells has at most two adjacent black unit square cells. Find the largest possible value of n.

Walkthrough —

(a)

Exercise 61 (Argentina National Olympiad 2020 Level 2 P2,). Let n be a positive integer. There are n colors available. Each of the integers from 1 to 1000 must be painted with one of the n colors such that any two different numbers, if one divides the other, are painted in different colors. Determine the smallest value of n for which this is possible.

Walkthrough —

(a)

Exercise 62 (Brazil National Olympiad 2022 Level 3 P6,). Some cells of a 10×10 chessboard are colored blue. A set of six cells is called *gremista* when the cells are the intersection of three rows and two columns, or two rows and three columns, and are painted blue. Determine the greatest value of n for which it is possible to color n chessboard cells blue such that there is not a gremista set.

Walkthrough —

(a)

Remark. This problem is equivalent to finding the maximum possible number of edges in a (possibly directed) graph on 10 vertices which is $K_{3,2}$-free.

Exercise 63 (Junior Balkan MO Shortlist 2022 C1,). Anna and Bob, with Anna starting first, alternately color the integers of the set $S = \{1, 2, \ldots, 2022\}$ red or blue. At their turn each one can color any uncolored number of S they wish with any color they wish. The game ends when all numbers of S get colored. Let N be the number of pairs (a, b), where a and b are elements of S, such that a, b have the same color, and $b - a = 3$. Anna wishes to maximize N. What is the maximum value of N that she can achieve regardless of how Bob plays?

Walkthrough —

(a)

Exercise 64 (Junior Balkan MO Shortlist 2022 C3,). There are 200 boxes on the table. In the beginning, each of the boxes contains a positive integer (the integers are not necessarily distinct). Every minute, Alice makes one move. A move consists of the following. First, she picks a box X which contains a number c such that $c = a + b$ for some numbers a and b which are contained in some other boxes. Then she picks a positive integer $k > 1$. Finally, she removes c from X and replaces it with kc. If she cannot make any moves, she stops. Prove that no matter how Alice makes her moves, she won't be able to make infinitely many moves.

Walkthrough —

(a)

Exercise 65 (Junior Balkan MO Shortlist 2023 C1,). Given is a square board with dimensions 2023×2023, in which each unit cell is colored blue or red. There are exactly 1012 rows in which the majority of cells are blue, and exactly 1012 columns in which the majority of cells are red. What is the maximal possible side length of the largest monochromatic square?

Walkthrough —

(a)

Exercise 66 (Canadian Junior Mathematical Olympiad 2023 P3, , cf. CMO 2023 P1)**.** William is thinking of an integer between 1 and 50, inclusive. Victor can choose a positive integer m and ask William: "does m divide your number?", to which William must answer truthfully. Victor continues asking these questions until he determines William's number. What is the minimum number of questions that Victor needs to guarantee this?

Walkthrough —

(a)

Exercise 67 (Rioplatense Mathematical Olympiad 2025 Level 3 Grade 12 P2,)**.** Let $n \geq 2$ be an integer. There are n numbers, each one starts at 0. In each operation, one chooses two numbers, the smallest increases by 1 and the biggest becomes 0 (if they are equal, one increases 1 and the other becomes 0). This process can be repeated indefinitely. Find, in function of n, the biggest number that can appear in the sequence.

Exercise 68 (All-Russian Mathematical Olympiad 2025 Grade 10 Day 2 P5, , also 11.5, proposed by I. I. Bogdanov)**.** Let n be a natural number. The numbers $1, 2, \ldots, n$ are written in a row in some order. For each pair of adjacent numbers, their greatest common divisor is calculated and written on a sheet. What is the maximum possible number of distinct values among the $n - 1$ integers obtained?

Walkthrough —

(a)

Remark. One may also observe that if ℓ denotes the largest GCD obtained from the adjacent numbers in the row, then it is the gcd of two distinct integers between 1 and n, and the larger of these two integers is at least 2ℓ, implying that $\ell \leq n/2$.

Exercise 69 (RMO 2016g P2,)**.** On a stormy night ten guests came to dinner party and left their shoes outside the room in order to keep the carpet clean. After the dinner there was a blackout, and the guests leaving one by one, put on at random, any pair of shoes big enough for their feet. (Each pair of shoes stays together). Any guest who could not find a pair big enough spent the night there. What is the largest number of guests who might have had to spend the night there?

> **Walkthrough** — What happens if a person having shoe of the smallest size wears a shoe of the largest size, and next, if a person having shoe of the second smallest size wears a shoe of the second largest size, and it continues?

Exercise 70 (RMO 2016c P4,). There are 100 countries participating in an olympiad. Suppose n is a positive integer such that each of the 100 countries is willing to communicate in exactly n languages. If each set of 20 countries can communicate in at least one common language, and no language is common to all 100 countries, what is the minimum possible value of n?

> **Walkthrough** —
>
> **(a)** Choose a country C, and let L_1 denote a language in which C communicates. Note that L_1 is not common to some country C_1. Hence, some language L_2 is common to the countries C, C_1.
>
> **(b)** The language L_2 is not common to some country C_2. Hence, some language L_3 is common to the countries C, C_1, C_2.
>
> **(c)** **Continuing** this line of argument, show that there are pairwise distinct languages $L_1, \ldots, L_{20}$ and C communicates in them.
>
> **(d)** Conclude that $n \geq 20$.
>
> **(e)** Show that 20 is the minimum possible value of n.

Exercise 71 (RMO 2018b P4,). Suppose 100 points in the plane are coloured using two colours, red and white, such that each red point is the centre of a circle passing through at least three white points. What is the least possible number of white points?

> **Summary** — It relies on the fact that one can find enough points on the plane such that no three of them are collinear and no four of them are concyclic.

> **Walkthrough** —
>
> **(a)** There is an upper bound on the number of the red points in terms of the number of the white points. This gives an upper bound on the total number of points, which is 100, in terms of the number of the white points.
>
> **(b)** Use this bound to guess the least possible number of the white points,

which would turn out to be 10.

(c) Begin with 10 white points on the plane in *general position*, and then, introduce enough red points to construct a configuration of 100 points with the stated properties.

Exercise 72 (INMO 2025 P5, , proposed by Pranjal Srivastava and Rohan Goyal)**.** Greedy goblin Griphook has a regular 2000-gon, whose every vertex has a single coin. In a move, he chooses a vertex, removes one coin each from the two adjacent vertices, and adds one coin to the chosen vertex, keeping the remaining coin for himself. He can only make such a move if both adjacent vertices have at least one coin. Griphook stops only when he cannot make any more moves. What is the maximum and minimum number of coins that he could have collected?

Walkthrough —

(a) Show that 1998 coins can be collected (see Fig. 14).

(b) Show that the difference of the number of coins at the vertices with odd indices, and the number of coins at the vertices with even indices, does not change modulo 3 after any move.

(c) Conclude that no more than 1998 coins can be collected.

(d) Show that 668 coins can be collected.

(e) Show that at least 668 coins are collected when Griphook stops.

The following is from AoPS, and is due to Rohan Goyal.

Exercise 73 (Moscow Mathematical Olympiad 2025 Grade 9 P4, proposed by A. Zaslavsky)**.** Each cell of a 100×100 square is painted either white or black. It turns out that each white cell has exactly two adjacent cells painted white, and each black cell has exactly two adjacent cells painted black. Find the maximum possible number of black cells.

Walkthrough —

(a)

Exercise 74 (All-Russian Mathematical Olympiad 2025 Grade 9 Day 2 P6, , proposed by S. L. Berlov)**.** Petya chooses 100 pairwise distinct positive numbers less than 1 and arranges them in a circle. In one operation, he may take three consecutive numbers a, b, c (in this order) and replace b with $a-b+c$. What is the greatest value of k such that Petya could initially choose the numbers and perform several operations so that k of the resulting numbers are integers?

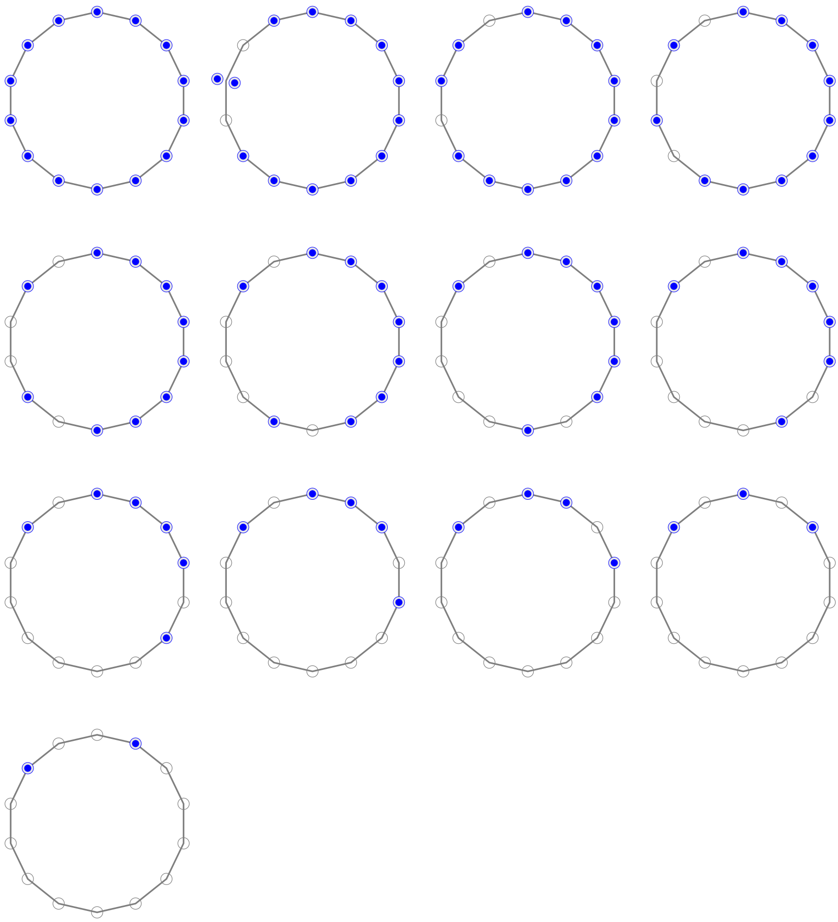


Figure 14: INMO 2025 P5, Collecting 12 coins, Exercise 72

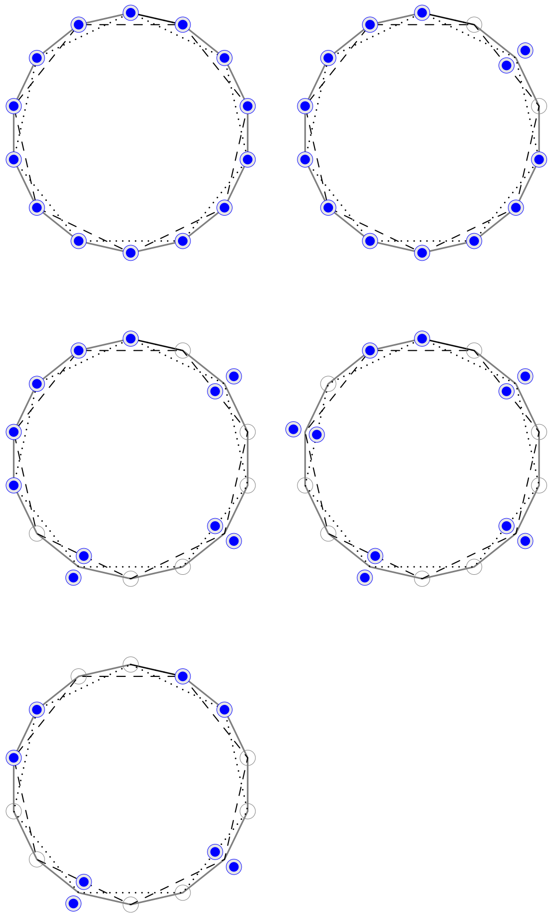


Figure 15: INMO 2025 P5, Collecting 6 coins, Exercise 72

Walkthrough —

(a)