Combinatorics

Regional Mathematical Olympiad Problems

1990–2025

§1 Warm up

Exercise 1 (RMO 2003 P7,). Consider the set $X = \{1, 2, 3, \ldots, 9, 10\}$. Find two disjoint nonempty subsets A and B of X such that

(a) $A \cup B = X$,

(b) prod(A) is divisible by prod(B), where for any finite set of numbers C, prod(C) denotes the product of all numbers in C,

(c) the quotient prod(A)/prod(B) is as small as possible.

Exercise 2 (RMO 2013c P6,). Let $n \geq 4$ be a natural number. Let $A_1A_2 \cdots A_n$ be a regular polygon and $X = \{1, 2, 3, \ldots, n\}$. A subset $\{i_1, i_2, \ldots, i_k\}$ of X, with $k \geq 3$ and $i_1 < i_2 < \cdots < i_k$, is called a *good subset* if the angles of the polygon $A_{i_1}A_{i_2} \cdots A_{i_k}$, when arranged in the increasing order, are in an arithmetic progression. If n is a prime, show that a proper good subset of X contains exactly four elements.

Exercise 3 (RMO 2014d P6,). In Fig. 1, can the numbers $1, 2, 3, 4, \ldots, 18$ be placed, one on each line segment, such that the sum of the numbers on the three line segments meeting at each point is divisible by 3?

Exercise 4 (RMO 2016g P1,). Suppose in a given collection of 2016 integers, the sum of any 1008 integers is positive. Show that sum of all 2016 integers is positive.

Exercise 5 (RMO 2017b P1). Consider a chessboard of size 8 units $\times$ 8 units (i.e. each small square on the board has a side length of 1 unit). Let S be the set of all the 81 vertices of all the squares on the board. What is the number of line segments whose vertices are in S, and whose length is a positive integer? (The segments need not be parallel to the sides of the board.)

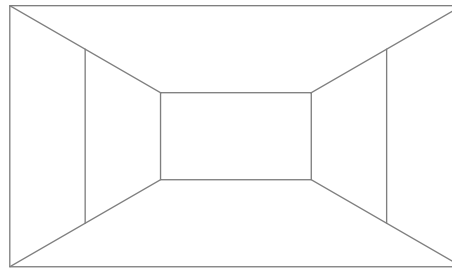


Figure 1: India RMO 2014d P6, Exercise 3

Exercise 6 (RMO 2017b P3)**.** Let $P(x)$ and $Q(x)$ be polynomials of degree 6 and degree 3 respectively, such that

$$P(x) > Q(x)^2 + Q(x) + x^2 - 6 \text{ for all } x \in \mathbb{R}.$$

If all the roots of $P(x)$ are real numbers, then prove that there exist two roots of $P(x)$, say α, β such that $|\alpha - \beta| < 1$.

Exercise 7 (RMO 2025b P1,)**.**

(a) Let $n \geq 3$ be an integer. Find a configuration of n lines in the plane which has exactly

(i) $n - 1$ distinct points of intersection;

(ii) n distinct points of intersection.

(b) Give configurations of n lines that have exactly $n + 1$ distinct points of intersection for (i) $n = 8$ and (ii) $n = 9$.

§2 Invariants

Exercise 8 (RMO 2014b P6,)**.** [**Eng98**, Problem 28, p. 29, p. 35] Let n be an odd positive integer and suppose that each square of an $n \times n$ grid is arbitrarily filled with either by 1 or by -1. Let r_j and c_k denote the product of all numbers in j-th row and k-th column respectively, $1 \leq j, k \leq n$. Prove that

$$\sum_{j=1}^{n} r_j + \sum_{k=1}^{n} c_k \neq 0.$$

Exercise 9 (RMO 2016d P4,)**.** A box contains 4032 answer scripts out of which exactly half have odd number of marks. We choose 2 scripts randomly and, if the scores on both of them are odd number, we add one mark to one of them, put the script back in the box and keep the other script outside. If both scripts have even scores, we put back one of the scripts and keep the other

outside. If there is one script with even score and the other with odd score, we put back the script with the odd score and keep the other script outside. After following this procedure a number of times, there are 3 scripts left among which there is at least one script each with odd and even scores. Find, with proof, the number of scripts with odd scores among the three left.

§3 Strategies

Exercise 10 (RMO 2016e P6,). A deck of 52 cards is given. There are four suites each having cards numbered $1, 2, \ldots, 13$. The audience chooses some five cards with distinct numbers written on them. The assistant of the magician comes by, looks at the five cards and turns exactly one of them face down and arranges all five cards in some order. Then the magician enters and with an agreement made beforehand with the assistant, he has to determine the face down card (both suite and number). Explain how the trick can be completed.

§4 Optimization problems

Exercise 11 (RMO 2016c P4,). There are 100 countries participating in an olympiad. Suppose n is a positive integer such that each of the 100 countries is willing to communicate in exactly n languages. If each set of 20 countries can communicate in at least one common language, and no language is common to all 100 countries, what is the minimum possible value of n?

Exercise 12 (RMO 2016g P2,). On a stormy night ten guests came to dinner party and left their shoes outside the room in order to keep the carpet clean. After the dinner there was a blackout, and the guests leaving one by one, put on at random, any pair of shoes big enough for their feet. (Each pair of shoes stays together). Any guest who could not find a pair big enough spent the night there. What is the largest number of guests who might have had to spend the night there?

Exercise 13 (RMO 2017a P4,). Consider n^2 unit squares in the xy-plane centred at the point (i, j) with integer coordinates for $1 \leq i \leq n, 1 \leq j \leq n$. It is required to colour each unit square in such a way that whenever $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$, the three squares with centres at (i, k), (j, k), (j, ℓ) have distinct colours. What is the least possible number of colours needed?

Exercise 14 (RMO 2018b P4,). Suppose 100 points in the plane are coloured using two colours, red and white, such that each red point is the centre of a circle passing through at least three white points. What is the least possible number of white points?

§5 Auxiliary configuration

Exercise 15 (RMO 1998 P6,). Given the 7-element set $A = \{a, b, c, d, e, f, g\}$, find a collection T of 3-element subsets of A such that each pair of elements from A occurs exactly in one of the subsets of T.

Exercise 16 (RMO 2023a P4,). The set X of N four-digit numbers formed from the digits $1, 2, 3, 4, 5, 6, 7, 8$ satisfies the following condition: *for any two different digits from* $1, 2, 3, 4, 5, 6, 7, 8$*, there exists a number in* X *which contains both of them.* Determine the smallest possible value of N.

§6 Induction

Exercise 17 (RMO 1991 P4,). There are two urns each containing an arbitrary number of balls. Both are non-empty to begin with. We are allowed two types of operations:

1. remove an equal number of balls simultaneously from the urns, and
2. double the number of balls in any one of them.

Show that after performing these operations finitely many times, both the urns can be made empty.

§7 Inclusion-exclusion principle

Exercise 18 (RMO 1993 P8,). I have 6 friends and during a vacation I met them during several dinners. I found that I dined with all the 6 exactly on 1 day; with every 5 of them on 2 days; with every 4 of them on 3 days; with every 3 of them on 4 days; with every 2 of them on 5 days. Further every friend was present at 7 dinners and every friend was absent at 7 dinners. How many dinners did I have alone?

Exercise 19 (RMO 2015f P6). From the list of natural numbers $1, 2, 3, \ldots$, suppose we remove all multiples of 7, 11 and 13.

- At which position in the resulting list does the number 1002 appear?
- What number occurs in the position 3600?

Exercise 20 (RMO 2019b P5,). There is a pack of 27 distinct cards, and each card has three values on it. The first value is a shape from $\{\Delta, \square, \odot\}$; the second value is a letter from $\{A, B, C\}$; and the third value is a number from $\{1, 2, 3\}$. In how many ways can we choose an unordered set of 3 cards from the pack, so that no two of the chosen cards have two matching values. For example we can chose $\{\Delta A1, \Delta B2, \odot C3\}$. But we cannot choose $\{\Delta A1, \square B2, \Delta C1\}$.

§8 Pigeonhole principle

Exercise 21 (RMO 1990 P1,). Two boxes contain between them 65 balls of several different sizes. Each ball is white, black, red or yellow. If you take any 5 balls of the same colour at least two of them will always be of the same size (radius). Prove that there are at least 3 balls which lie in the same box have the same colour and have the same size (radius).

Exercise 22 (RMO 1996 P7,). If A is a fifty-element subset of the set $\{1, 2, 3, \ldots, 100\}$ such that no two numbers from A add up to 100, show that A contains a square.

Exercise 23 (RMO 2011a P2,). Let $(a_1, a_2, a_3, \ldots, a_{2011})$ be a permutation of the numbers $1, 2, 3, \ldots, 2011$. Show that there exist two numbers j, k such that $1 \leq j < k \leq 2011$ and $|a_j - j| = |a_k - k|$.

Exercise 24 (RMO 2013d P6,). Suppose that the vertices of a regular polygon of 20 sides are coloured with three colors red, blue and green, such that there are exactly three red vertices. Prove that there are three vertices A, B, C of the polygon having the same colour such that triangle ABC is isosceles.

Exercise 25 (RMO 2014a P4,). Is it possible to write the numbers $17, 18, 19, \ldots, 32$ in a 4×4 grid of unit squares with one number in each square such that if the grid is divided into four 2×2 subgrids of unit squares, then the product of numbers in each of the subgrids divisible by 16?

Exercise 26 (RMO 2018a P4,). Let E denote the set of 25 points (m, n) in the xy-plane, where m, n are natural numbers, $1 \leq m \leq 5, 1 \leq n \leq 5$. Suppose the points of E are arbitrarily coloured using two colours, red and blue. Show that there always exist four points in the set E of the form $(a, b), (a + k, b), (a + k, b + k), (a, b + k)$ for some positive integer k such that at least three of these four points have the same colour. (That is, there always exist four points in the set E which form the vertices of a square and having at least three points of the same colour.)

Exercise 27 (RMO 2019b P4,). Let $a_1, a_2, \ldots, a_6, a_7$ be seven positive integers. Let S be the set of all numbers of the form $a_i^2 + a_j^2$ where $1 \leq i < j \leq 7$. Prove that there exist two elements of S which have the same remainder on dividing by 36.

Exercise 28 (RMO 2023b P6,). Consider a set of 16 points arranged in a 4×4 square grid formation. Prove that if any 7 of these points are coloured blue, then there exists an isosceles right-angled triangle whose vertices are all blue.

Exercise 29 (RMO 2024a P6,). Let X be a set of 11 integers. Prove that one can find a nonempty subset $\{a_1, a_2, \ldots, a_k\}$ of X such that 3 divides k and 9 divides the sum $\sum_{i=1}^{k} 4^i a_i$.

§9 Extremal principle

Exercise 30 (RMO 1991 P8,). The 64 squares of an 8×8 chessboard are filled with positive integers in such a way that each integer is the average of the integers on the neighbouring squares. (Two squares are neighbours if they share a common edge or a common vertex. Thus a square can have $8, 5$ or 3 neighbours depending on its position). Show that all the 64 integer entries are in fact equal.

§10 Arrange in Order

Exercise 31 (RMO 2002 P4,). Suppose the integers $1, 2, 3, \ldots, 10$ are split into two disjoint collections a_1, a_2, a_3, a_4, a_5 and b_1, b_2, b_3, b_4, b_5 such that $a_1 < a_2 < a_3 < a_4 < a_5$ and $b_1 > b_2 > b_3 > b_4 > b_5$.

(i) Show that the larger number in any pair $\{a_j, b_j\}$, $1 \leq j \leq 5$, is at least 6.

(ii) Show that $|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5| = 25$ for every such partition.

Exercise 32 (RMO 2006 P4,). A 6×6 square is dissected into 9 rectangles by lines parallel to its sides such that all these rectangles have only integer sides. Prove that there are always two congruent rectangles.

§11 Grouping in pairs

Exercise 33 (RMO 1994 P7,). Find the number of all rational numbers m/n such that

1. $0 < m/n < 1$,
2. m and n are relatively prime,
3. $mn = 25!$.

Exercise 34 (RMO 2012f P6,). Let S be the set $\{1, 2, \ldots, 10\}$. Let A be a subset of S. We arrange the elements of A in increasing order, that is, $A = \{a_1, a_2, \ldots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$. Define WSUM for this subset as $3(a_1 + a_3 + \ldots) + 2(a_2 + a_4 + \ldots)$ where the first term contains the odd numbered terms and the second the even numbered terms. (For example, if $A = \{2, 5, 7, 8\}$, WSUM is $3(2 + 7) + 2(5 + 8)$.) Find the sum of WSUMs over all the subsets of S. (Assume that WSUM for the null set is 0.)

§12 Counting

Exercise 35 (RMO 1993 P3,). Suppose $A_1A_2\cdots A_{20}$ is a 20-sided regular polygon. How many non-isosceles (scalene) triangles can be formed whose vertices are among the vertices of the polygon but whose sides are not the sides of the polygon?

Exercise 36 (RMO 1995 P6,). Suppose $A_1A_2\cdots A_{21}$ is a 21-sided regular polygon inscribed in a circle with centre O. How many triangles $A_iA_jA_k$, $1 \le i < j < k \le 21$, contain the centre point O in their interior?

Exercise 37 (RMO 2000 P4,). All the 7-digit numbers containing each of the digits $1, 2, 3, 4, 5, 6, 7$ exactly once, and not divisible by 5, are arranged in increasing order. Find the 2000-th number in this list.

Exercise 38 (RMO 2001 P4,). Consider an $n \times n$ array of numbers

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}.$$

Suppose each row consists of the n numbers $1, 2, \dots, n$ in some order and $a_{ij} = a_{ji}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$. If n is odd, prove that the numbers $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are $1, 2, \dots, n$ in some order.

Exercise 39 (RMO 2005 P4,). Find the number of all 5-digit numbers (in base 10) each of which contains the block 15 and is divisible by 15. (For example, $31545, 34155$ are two such numbers.)

Exercise 40 (RMO 2007 P4,). How many 6-digit numbers are there such that

(a) the digits of each number are all from the set $\{1, 2, 3, 4, 5\}$,

(b) any digit that appears in the number appears at least twice?

For example, 225252 is valid while 222133 is not.

Exercise 41 (RMO 2008 P4,). Find the number of all 6-digit natural numbers such that the sum of their digits is 10 and each of the digits $0, 1, 2, 3$ occurs at least once in them.

Exercise 42 (RMO 2010 P3,). Find the number of 4-digit numbers (in base 10) having non-zero digits and which are divisible by 4 but not by 8.

Exercise 43 (RMO 2011a P4,). Consider a 20-sided convex polygon K, with vertices $A_1, A_2, \ldots, A_{20}$ in that order. Find the number of ways in which three sides of K can be chosen so that every pair among them has at least two sides of K between them. (For example $(A_1A_2, A_4A_5, A_{11}A_{12})$ is an admissible triple while $(A_1A_2, A_4A_5, A_{19}A_{20})$ is not.)

Exercise 44 (RMO 2011b P4,). Find the number of 4-digit numbers with distinct digits chosen from the set $\{0, 1, 2, 3, 4, 5\}$ in which no two adjacent digits are even.

Exercise 45 (RMO 2013c P1,). Find the number of eight-digit numbers the sum of whose digits is 4.

Exercise 46 (RMO 2013e P3,). A finite non-empty set of integers is called 3-*good* if the sum of its elements is divisible by 3. Find the number of 3-good subsets of $\{0, 1, 2, \ldots, 9\}$.

Exercise 47 (RMO 2014e P4,). A person moves in the x-y plane moving along points with integer co-ordinates x and y only. When she is at a point (x, y), she takes a step based on the following rules:

(a) if $x + y$ is even she moves to either $(x + 1, y)$ or $(x + 1, y + 1)$,

(b) if $x + y$ is odd she moves to either $(x, y + 1)$ or $(x + 1, y + 1)$.

How many distinct paths can she take to go from $(0, 0)$ to $(8, 8)$ given that she took exactly three steps to the right ((x, y) to $(x + 1, y)$)?

Exercise 48 (RMO 2015a P4, , cf. RMO 2015b P4, RMO 2015d P4, RMO 2015e P4). Suppose 36 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

Exercise 49 (RMO 2015b P4, , cf. RMO 2015a P4, RMO 2015d P4, RMO 2015e P4). Suppose 40 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

Exercise 50 (RMO 2015c P2,). Determine the number of 3 digit numbers in base 10 having at least one digit equal to 5 and at most one digit equal to 3.

Exercise 51 (RMO 2015d P4, , cf. RMO 2015a P4, RMO 2015b P4, RMO 2015e P4). Suppose 32 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

Exercise 52 (RMO 2015e P4, , cf. RMO 2015a P4, RMO 2015b P4, RMO 2015d P4). Suppose 28 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

Exercise 53 (RMO 2016a P4,). Find the number of all 6-digit natural numbers having exactly three odd digits and three even digits.

Exercise 54 (RMO 2016b P4,). How many 6-digit natural numbers containing only the digits $1, 2, 3$ are there in which 3 occurs exactly twice and the number is divisible by 9?

Exercise 55 (RMO 2016f P2,). At an international event there are 100 countries participating, each with its own flag. There are 10 distinct flagpoles at the stadium, labelled $\#1, \#2, \ldots, \#10$ in a row. In how many ways can all the 100 flags be hoisted on these 10 flagpoles, such that for each i from 1 to 10, the flagpole $\#i$ has at least i flags? (Note that the vertical order of the flagpoles on each flag is important)

Exercise 56 (RMO 2025a P4,). A frog is initially at $(0, 0)$ and it reaches $(n, 2)$, $n \geq 1$, using the following moves in any order several times:

1. $R = (1, 0)$, that is, if the frog is at (a, b) it goes to $(a + 1, b)$;
2. $U = (0, 1)$, that is, if the frog is at (a, b) it goes to $(a, b + 1)$;
3. $D = (1, 1)$, that is, if the frog is at (a, b) it goes to $(a + 1, b + 1)$.

In how many ways can the frog go from $(0, 0)$ to $(n, 2)$, $n \geq 1$, using the above steps subject to the condition that steps of the type UU, DD are forbidden? (For example, for $n = 3$, $RDUR, DRD$ are admissible paths, while $DDR, RUURR$ are not.)

§13 Counting via bijections

Exercise 57 (RMO 1997 P6,). Find the number of unordered pairs $\{A, B\}$ (that is, the pairs $\{A, B\}$ and $\{B, A\}$ are considered to be the same) of subsets of an n-element set X which satisfy the conditions:

(a) $A \neq B$,

(b) $A \cup B = X$.

(eg., if $X = \{a, b, c, d\}$, then $\{\{a, b\}, \{b, c, d\}\}$, $\{\{a\}, \{b, c, d\}\}$, $\{\emptyset, \{a, b, c, d\}\}$ are some of the admissible pairs.)

Exercise 58 (RMO 2012a P4, , cf. RMO 2012b P4, RMO 2012c P4, RMO 2012d P4). Let $X = \{1, 2, 3, \ldots, 10\}$. Find the number of pairs $\{A, B\}$ such that $A \subseteq X, B \subseteq X, A \neq B$ and $A \cap B = \{2, 3, 5, 7\}$.

Exercise 59 (RMO 2012b P4, , cf. RMO 2012a P4, RMO 2012c P4, RMO 2012d P4). Let $X = \{1, 2, 3, \ldots, 12\}$. Find the number of pairs $\{A, B\}$ such that $A \subseteq X, B \subseteq X, A \neq B$ and $A \cap B = \{2, 3, 5, 7, 8\}$.

Exercise 60 (RMO 2012c P4, cf. RMO 2012a P4, RMO 2012b P4, RMO 2012d P4)**.** Let $X = \{1, 2, 3, \ldots, 10\}$. Find the number of pairs of $\{A, B\}$ such that $A \subseteq X, B \subseteq X, A \neq B$ and $A \cap B = \{5, 7, 8\}$.

Exercise 61 (RMO 2012d P4, cf. RMO 2012a P4, RMO 2012b P4, RMO 2012c P4)**.** Let $X = \{1, 2, 3, \ldots, 11\}$. Find the number of pairs of $\{A, B\}$ such that $A \subseteq X, B \subseteq X, A \neq B$ and $A \cap B = \{4, 5, 7, 8, 9, 10\}$.

Exercise 62 (RMO 2013b P6)**.** For a natural number n, let $T(n)$ denote the number of ways we can place n objects of weights $1, 2, \ldots, n$ on a balance such that the sum of the weights in each pan is the same. Prove that $T(100) > T(99)$.

Exercise 63 (RMO 2015c P6)**.** Let $S = \{1, 2, \ldots, n\}$ and let T be the set of all ordered triples of subsets of S, say (A_1, A_2, A_3), such that $A_1 \cup A_2 \cup A_3 = S$. Determine in terms of n,

$$\sum_{(A_1, A_2, A_3) \in T} |A_1 \cap A_2 \cap A_3|,$$

where $|X|$ denotes the number of elements in the set X.

Exercise 64 (RMO 2024b P6)**.** Let $n \geq 2$ be a positive integer. Call a sequence $a_1, a_2, \ldots, a_k$ of integers an *n-chain* if $1 = a_1 < a_2 < \cdots < a_k = n$, a_i divides a_{i+1} for all i, $1 \leq i \leq k - 1$. Let $f(n)$ be the number of n-chains where $n \geq 2$. For example, $f(4) = 2$ corresponds to the 4-chains $\{1, 4\}$ and $\{1, 2, 4\}$. Prove that $f(2^m \cdot 3) = 2^{m-1}(m + 2)$ for every positive integer m.

§14 Counting the complement

Exercise 65 (RMO 2004 P4)**.** Prove that the number of triples (A, B, C) where A, B, C are subsets of $\{1, 2, \ldots, n\}$ such that $A \cap B \cap C = \emptyset$, $A \cap B \neq \emptyset$, $B \cap C \neq \emptyset$ is $7^n - 2 \cdot 6^n + 5^n$.

Exercise 66 (RMO 2009 P4)**.** Find the sum of all 3-digit natural numbers which contain at least one odd digit and at least one even digit.

§15 Counting in two different ways

Exercise 67 (RMO 2019a P6)**.** Suppose 91 distinct positive integers greater than 1 are given such that there are at least 456 pairs among them which are relatively prime. Show that one can find four integers a, b, c, d among them such that $\gcd(a, b) = \gcd(b, c) = \gcd(c, d) = \gcd(d, a) = 1$.

Exercise 68 (RMO 2023a P4)**.** The set X of N four-digit numbers formed from the digits $1, 2, 3, 4, 5, 6, 7, 8$ satisfies the following condition: *for any two different digits from $1, 2, 3, 4, 5, 6, 7, 8$, there exists a number in X which contains both of them.* Determine the smallest possible value of N.

References

[**Eng98**] Arthur Engel. *Problem-solving strategies*. Problem Books in Mathematics. Springer-Verlag, New York, 1998, pp. x+403. ISBN: 0-387-98219-1 (cited p. 2)