§1 Warm up

Example 1.1 (India RMO 2018)**.** Suppose 100 points in the plane are coloured using two colours, red and white, such that each red point is the centre of a circle passing through at least three white points. What is the least possible number of white points?

Summary — It relies on the fact that one can find enough points on the plane such that no three of them are collinear and no four of them are concyclic.

Walkthrough —

(a) There is an upper bound on the number of the red points in terms of the number of the white points. This gives an upper bound on the total number of points, which is 100, in terms of the number of the white points.

(b) Use this bound to guess the least possible number of the white points, which would turn out to be 10.

(c) Begin with 10 white points on the plane in *general position*, and then, introduce enough red points to construct a configuration of 100 points with the stated properties.

Solution 1. Let n denote the number of white points. Since each red point is the centre of a circle passing through at least three white points, it follows that the number of red points is at most $\binom{n}{3}$. This shows that

$$n + \binom{n}{3} \geq 100.$$

Note that $n \mapsto n + \binom{n}{3}$ defines an increasing function on the nonnegative integers. Observe that

$$9 + \binom{9}{3} = 93, \quad 10 + \binom{10}{3} = 130.$$

This implies that $n \geq 10$.

We claim that there is a configuration of 100 points on the plane such that it admits a coloring using two colors, red and white, such that precisely 10 points are colored white, and that each red point is the centre of a circle passing through at least three white points. Indeed, consider 10 points on the plane such that no three of them are collinear and no four of them are concyclic [1]. Color these 10 points white. These white points have $\binom{10}{3} = 120$ subsets of

[1] Why does such a collection exist? This could be intuitively clear, but can you write down a precise proof? Does induction help?

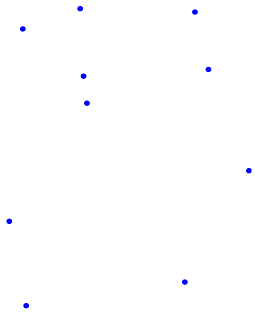

Figure 1: A set of 10 points, Example 1.1

size 3. Consider only 90 such subsets of the white points, and for any such subset of size 3, color the center of the circle passing through them red. Since no three white points are collinear and no four white points are concyclic, it follows that there are precisely 90 pairwise distinct red points. So, the red and the white points together form a set of 100 points such that each red point is the centre of a circle passing through at least three white points. ■

In Fig. 1, there are 10 blue points. In Fig. 2, there are 10 blue points and the circumcenters of their 3-subsets. In Fig. 3, there are 10 blue points together with the circumcenters and circumcircles of their 3-subsets.

List of problems and examples

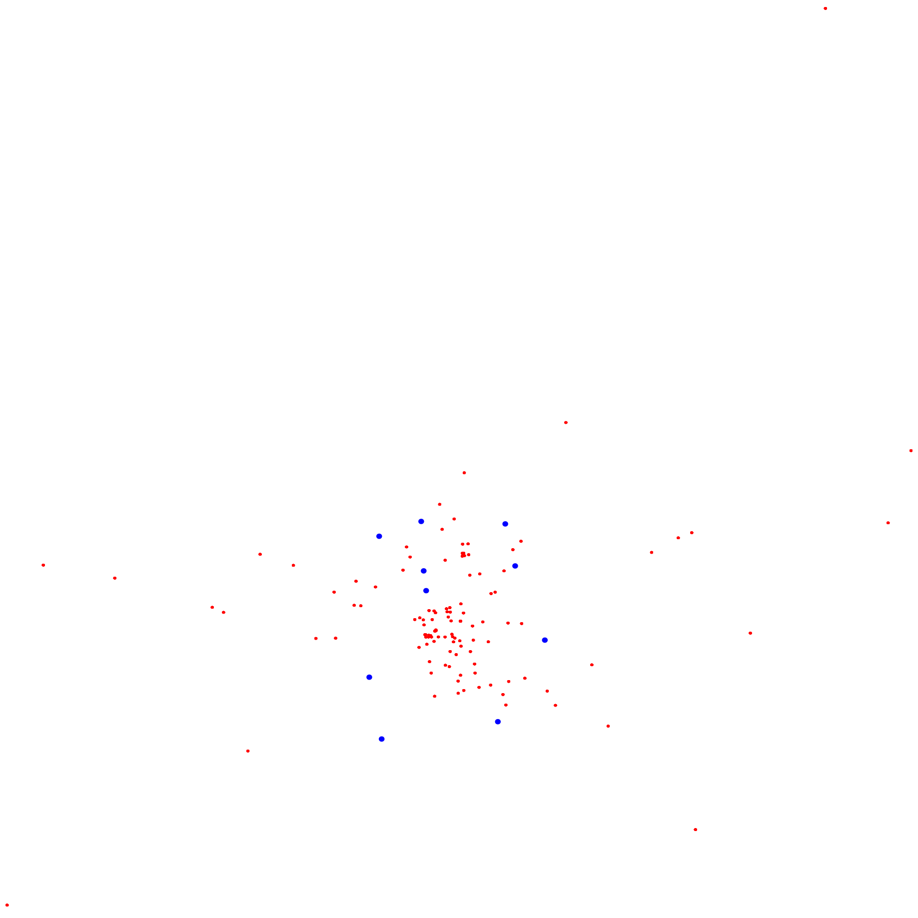

Figure 2: A set of 10 points and the circumcenters of their 3-subsets, Example 1.1

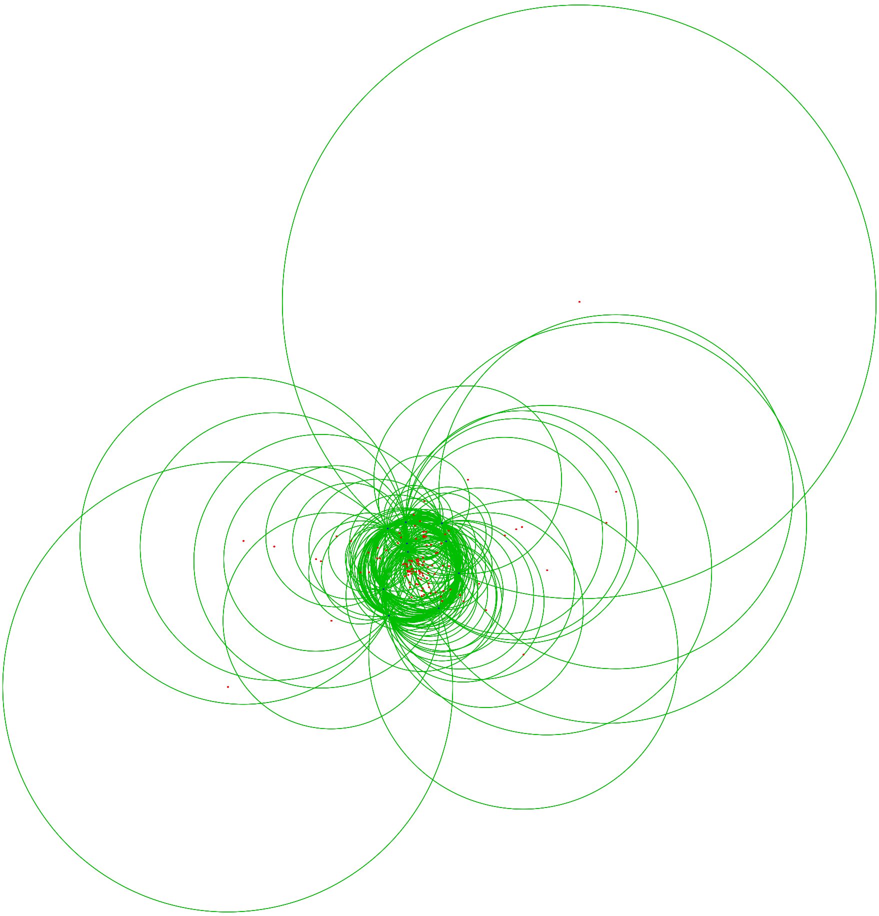

Figure 3: A set of 10 points together with the circumcenters and circumcircles of their 3-subsets, Example 1.1